

THE abc -PROBLEM FOR GABOR SYSTEMS

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ABSTRACT. A Gabor system generated by a window function ϕ and a rectangular lattice $a\mathbb{Z} \times \mathbb{Z}/b$ is given by

$$\mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z}/b) := \{e^{-2\pi i n t/b} \phi(t - ma) : (m, n) \in \mathbb{Z} \times \mathbb{Z}\}.$$

One of fundamental problems in Gabor analysis is to identify window functions ϕ and time-frequency shift lattices $a\mathbb{Z} \times \mathbb{Z}/b$ such that the corresponding Gabor system $\mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame for $L^2(\mathbb{R})$, the space of all square-integrable functions on the real line \mathbb{R} . The range of density parameters a and b such that the Gabor system $\mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z}/b)$ is a frame for $L^2(\mathbb{R})$ is fully known surprisingly only for few window functions, including the Gaussian window and totally positive windows. Janssen's tie suggests that the range of density parameters could be arbitrarily complicated for window functions, especially outside Feichtinger algebra. An eye-catching example of such a window function is the ideal window function on an interval. In this paper, we provide a full classification of triples (a, b, c) for which the Gabor system $\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z}/b)$ generated by the ideal window function χ_I on an interval I of length c is a Gabor frame for $L^2(\mathbb{R})$. For the classification of such triples (a, b, c) (i.e., the abc -problem for Gabor systems), we introduce maximal invariant sets of some piecewise linear transformations and establish the equivalence between Gabor frame property and triviality of maximal invariant sets. We then study dynamic system associated with the piecewise linear transformations and explore various properties of their maximal invariant sets. By performing holes-removal surgery for maximal invariant sets to shrink and augmentation operation for a line with marks to expand, we finally parameterize those triples (a, b, c) for which maximal invariant sets are trivial. The novel techniques involving non-ergodicity of dynamical systems associated with some novel non-contractive and non-measure-preserving transformations lead to our arduous answer to the abc -problem for Gabor systems.

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1. INTRODUCTION

Let $L^2 := L^2(\mathbb{R})$ be the space of all square-integrable functions on the real line \mathbb{R} with the inner product and norm on L^2 denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2$ respectively. In this paper, a *Gabor system* generated by a window function $\phi \in L^2$ and a rectangular lattice $a\mathbb{Z} \times \mathbb{Z}/b$ is given by

$$(1.1) \quad \mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z}/b) := \{e^{-2\pi i n t/b} \phi(t - ma) : (m, n) \in \mathbb{Z} \times \mathbb{Z}\}$$

([4, 16, 17, 20]); a *frame* for L^2 is a collection \mathcal{F} of functions in L^2 satisfying

$$(1.2) \quad 0 < \inf_{\|f\|_2=1} \left(\sum_{\phi \in \mathcal{F}} |\langle f, \phi \rangle|^2 \right)^{1/2} \leq \sup_{\|f\|_2=1} \left(\sum_{\phi \in \mathcal{F}} |\langle f, \phi \rangle|^2 \right)^{1/2} < \infty$$

([5, 7, 8, 13]); and a *Gabor frame* is a Gabor system that forms a frame for L^2 ([21, 24]). Here we use $1/b$ to label the frequency spacing instead of b in other sources on Gabor theory, due to the convenience to state our full classification of triples (a, b, c) for which the Gabor system $\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z}/b)$ generated by the ideal window function χ_I on an interval I of length c is a Gabor frame for L^2 .

Gabor frames have links to operator algebra and complex analysis, and they have been shown to be suitable for lots of applications involving time-dependent frequency. The history of the Gabor theory could date back to the completeness claim in 1932 by von Neumann [30, p. 406] and the expansion conjecture in 1946 by Gabor [15, Eq. 1.29], with both confirmed to be correct in later 1970. Gabor frames become widely studied after the landmark paper in 1986 by Daubechies, Grossmann and Meyer, where they proved that for any positive density parameters a, b satisfying $a/b < 1$ there exists a compactly supported smooth function ϕ such that $\mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame [12].

One of fundamental problems in Gabor analysis is to identify window functions and time-frequency shift sets such that the corresponding Gabor system is a Gabor frame. Given a window function $\phi \in L^2$ and a rectangular lattice $a\mathbb{Z} \times \mathbb{Z}/b$, a well-known necessary condition for the Gabor system $\mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z}/b)$ to be a Gabor frame, obtained via Banach algebra technique, is that the density parameters a and b satisfy $a/b \leq 1$ [3, 9, 23, 28, 31]. But that necessary condition on density parameters is far from providing an answer to the above fundamental problem.

The range of density parameters a, b such that the Gabor system $\mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z}/b)$ is a frame for L^2 is fully known stunningly only for small number of window functions ϕ [18, 26, 27, 29, 33, 34]. Among those, the Gaussian window $\sqrt[4]{2}\exp(-\pi t^2)$ has received special attention [15, 30]. It is conjectured by Daubechies and Grossmann [11] and later proved independently by Lyubarskii [29] and by Seip and Wallsten [33, 34] via complex analysis technique that the Gabor system $\mathcal{G}(\sqrt[4]{2}\exp(-\pi t^2), a\mathbb{Z} \times \mathbb{Z}/b)$ associated with the Gaussian window is a Gabor frame if and only if $a/b < 1$. A significant advance on the range of density parameters was recently made by Gröchenig and Stöckler that for a totally positive function ϕ of finite type, $\mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if $a/b < 1$ [18].

For a window function ϕ in Feichtinger's algebra, an important result proved by Feichtinger and Kaiblinger [14] states that the range of density parameters a, b such that the Gabor system $\mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z}/b)$ is a frame for L^2 is an open domain on the plane. But for window functions outside the Feichtinger algebra, the range of density parameters could

be arbitrarily complicated, c.f. the famous Janssen's tie [25]. A striking example of such a window function is the ideal window function on an interval [6, 19, 25].

Denote by χ_E the characteristic function on a set E . Recall that given an interval I , $\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if $\mathcal{G}(\chi_{I+d}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame for every $d \in \mathbb{R}$. By the above shift-invariance, the interval I can be assumed to be half-open and have zero as its left endpoint, i.e., $I = [0, c)$ for some $c > 0$. Thus the range problem for the ideal window on an interval reduces to the so-called *abc*-problem for Gabor systems: *the classification of all triples (a, b, c) of positive numbers such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame* [6]. In this paper, we provide a complete answer to the above *abc*-problem for Gabor systems, see Theorems 2.1–2.5.

By dilation-invariance, $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if $\mathcal{G}(\chi_{[0,dc)}, da\mathbb{Z} \times \mathbb{Z}/(db))$ is for any $d > 0$. Thus the *abc*-problem for Gabor systems can be reduced to finding out all pairs (a, b) of time-frequency spacing parameters such that $\mathcal{G}(\chi_{[0,1)}, a\mathbb{Z} \times \mathbb{Z}/b)$ are Gabor frames [19, 25], or all pairs (a, c) of time-spacing and window-size parameters such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are Gabor frames. In this paper, we state our results without any normalization on any one of the time-spacing, frequency-spacing and window-size parameters as it does not help us much.

Our answer to the *abc*-problem for Gabor system is illustrated in Figure 1, where on the left subfigure we normalize the frequency-spacing parameter b , while on the right subfigure we normalize the window-size parameter c and use the conventional frequency-spacing parameter $1/b$ as the y -axis, c.f. Janssen's tie in [25]. For pairs (a, c) in the red region of the left subfigure of Figure 1, $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are not Gabor frames by Theorem 2.1, while for pairs (a, c) in the green, blue and dark blue regions $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are Gabor frames by Theorems 2.1–2.3. In the yellow region, it follows from Conclusion (VI) of Theorem 2.2 that the set of pairs (a, c) such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are not Gabor frames contains needles (line segments) of lengths $\gcd(\lfloor c \rfloor + 1, p)/q - \{0, 1/q\}$ hanging vertically from the ceiling $\lfloor c \rfloor + 1$ at every rational time shift location $a = p/q$. In the purple region, we obtain from Conclusion (VII) of Theorem 2.2 that the set of pairs (a, c) such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are not Gabor frames contains floors $\lfloor c \rfloor \geq 2$ and also needles (line segments) of lengths $\gcd(\lfloor c \rfloor, p)/q - \{0, 1/q\}$ growing vertically from floors $\lfloor c \rfloor$ at every rational time shift location $a = p/q$. It has rather complicated geometry for the set of pairs (a, c) in the white region such that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z})$ are not Gabor frames. That set contains some

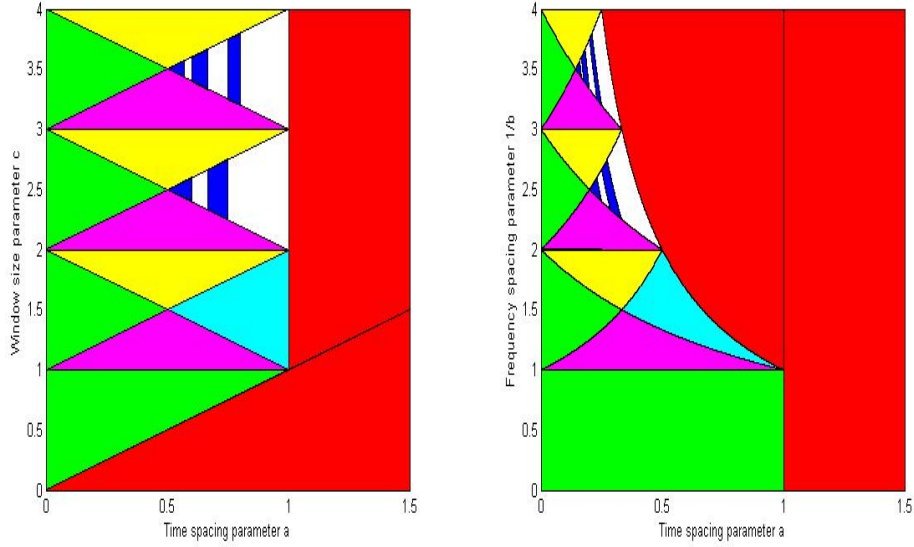


FIGURE 1. Left: Classification of pairs (a, c) such that $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z})$ are Gabor frames. Right: Classification of pairs $(a, 1/b)$ such that $\mathcal{G}(\chi_{[0,1]}, a\mathbb{Z} \times \mathbb{Z}/b)$ are Gabor frames.

needles (line segments) on the vertical line growing from rational time shift locations and few needle holes (points) on the vertical line located at irrational time shifts by Theorems 2.4 and 2.5. From the above observations, we see that the range of density parameters a, b such that the Gabor system $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ are Gabor frames is neither open nor connected, and it has very puzzling structure, c.f. the openness of the range of density parameters for a window function in Feichtinger's algebra [14].

The paper is organized as follows. In Section 2, we state our main theorems of this paper. The first two main theorems (Theorems 2.2 and 2.3) are proved in Sections 3 and 4 respectively. After studying various properties of the dynamic system associated with some non-contractive and non-measure-preserving transformations in Sections 5, we parameterize those triples (a, b, c) of positive numbers such that $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ are Gabor frames, and finally establish our most challenging results (Theorems 2.4 and 2.5) in Sections 6 and 7. In Appendix A, we provide a finite-step algorithm to verify whether the Gabor system $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame for any given triple of (a, b, c) of positive numbers. In Appendix B, we apply our results on Gabor systems to identify all intervals I and time-sampling spacing

lattices $b\mathbb{Z} \times a\mathbb{Z}$ such that signals f in the shift-invariant space

$$V_2(\chi_I, b\mathbb{Z}) = \left\{ \sum_{\lambda \in b\mathbb{Z}} d(\lambda) \chi_I(\cdot - \lambda) : \sum_{\lambda \in b\mathbb{Z}} |d(\lambda)|^2 < \infty \right\}$$

can be stably recovered from their equally-spaced samples $f(t_0 + \mu)$, $\mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$. In Appendix C, we discuss non-ergodicity of a new non-contractive and non-measure-preserving transformation.

In this paper, we will use the following notation. We denote the set of rational numbers by \mathbb{Q} ; the integral part of a real number t by $[t]$; the sign of a real number t by $\text{sgn}(t)$; the greatest common divisor between p and q in a lattice $r\mathbb{Z}$ with $r > 0$ by $\gcd(p, q)$; the Lebesgue measure of a measurable set E by $|E|$; the transpose of a matrix (vector) \mathbf{A} by \mathbf{A}^T ; the null space of a matrix \mathbf{A} by $N(\mathbf{A})$; the column vector whose entries take value $r \in \mathbb{R}$ by $\mathbf{r} := (\cdots, r, r, r, \cdots)^T$; and the space of all square-summable vectors $\mathbf{z} := (\mathbf{z}(\lambda))_{\lambda \in \Lambda}$ on a given index set Λ by $\ell^2 := \ell^2(\Lambda)$, with its standard norm and inner product denoted by $\|\cdot\|_2 := \|\cdot\|_{\ell^2(\Lambda)}$ and $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\ell^2(\Lambda)}$ respectively. Also for $b > 0$, we let $\mathcal{B}_b := \{(\mathbf{x}(\lambda))_{\lambda \in b\mathbb{Z}} \mid \mathbf{x}(\lambda) \in \{0, 1\} \text{ for all } \lambda \in b\mathbb{Z}\}$ consist of all *binary column vectors* whose components taking values either zero or one, and $\mathcal{B}_b^0 := \{(\mathbf{x}(\lambda))_{\lambda \in b\mathbb{Z}} \in \mathcal{B}_b \mid \mathbf{x}(0) = 1\}$ contain all binary vectors taking value one at the origin.

2. MAIN THEOREMS

In this section, we present our answer to the *abc*-problem for Gabor systems, and a confirmation to the conjecture in [25, Section 3.3.5].

Let us start from recalling some known classification of triples (a, b, c) of positive numbers, see for instance [12, 19, 25].

Theorem 2.1. *Let (a, b, c) be a triple of positive numbers, and let $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ be the Gabor system in (1.1) generated by the characteristic function on the interval $[0, c)$. Then the following statements hold.*

- (I) *If $a > c$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame.*
- (II) *If $a = c$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if $a \leq b$.*
- (III) *If $a < c$ and $b \leq a$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame.*
- (IV) *If $a < c$ and $b \geq c$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame.*

The conclusions in Theorem 2.1 are illustrated in the red and low right-triangle green regions of Figure 1.

By Theorem 2.1, it remains to classify triples (a, b, c) of positive numbers such that $a < b < c$. To do so, for any given triple (a, b, c) of positive numbers, we let

$$(2.1) \quad \mathbf{M}_{a,b,c}(t) := (\chi_{[0,c)}(t - \mu + \lambda))_{\mu \in a\mathbb{Z}, \lambda \in b\mathbb{Z}}, \quad t \in \mathbb{R},$$

and we introduce a periodic set

$$(2.2) \quad \mathcal{D}_{a,b,c} := \{t \in \mathbb{R} \mid \mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{2} \text{ for some binary vectors } \mathbf{x} \in \mathcal{B}_b^0\}$$

that contains all t on the real line such that there exists a binary vector solution $\mathbf{x} \in \mathcal{B}_b^0$ to the infinite-dimensional linear system

$$(2.3) \quad \mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{2}.$$

The uniform stability of infinite matrices $\mathbf{M}_{a,b,c}(t), t \in \mathbb{R}$, in (2.1),

$$(2.4) \quad 0 < \inf_{t \in \mathbb{R}} \inf_{\|z\|_2=1} \|\mathbf{M}_{a,b,c}(t)\mathbf{z}\|_2 \leq \sup_{t \in \mathbb{R}} \sup_{\|z\|_2=1} \|\mathbf{M}_{a,b,c}(t)\mathbf{z}\|_2 < \infty,$$

was used by Ron and Shen [32] to characterize the frame property for the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$, see Lemma 3.1 in Section 3.

We observe that infinite matrices $\mathbf{M}_{a,b,c}(t), t \in \mathbb{R}$, in (2.1) have their rows containing $\lfloor c/b \rfloor + \{0, 1\}$ consecutive ones, and their rows are obtained by shifting one (or zero) unit of the previous row with possible reduction or expansion by one unit. In the case that a/b is rational, infinite matrices in (2.1) have certain shift-invariance in the sense that their $(\mu + qa)$ -th row can be obtained by shifting p -units of the μ -th row where p and q are coprime integers satisfying $a/b = p/q$, c.f. [25, Eq. 3.3.68]. The above observations could be illustrated from examples below:

(2.5)

$$\mathbf{M}_{a,b,c}(0) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ & & & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ & & & & 0 & 1 & 1 & 1 & 1 & 0 \\ & & & & & 0 & 1 & 1 & 1 & 1 & 0 \\ & & & & & & 0 & 1 & 1 & 1 & 1 & 0 \\ & & & & & & & 0 & 1 & 1 & 1 & 1 & 0 \\ & & & & & & & & 0 & 1 & 1 & 1 & 1 & 0 \\ & & & & & & & & & 0 & 1 & 1 & 1 & 1 & 0 \\ & & & & & & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(2.6)

[illegible]

(2.7) $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if $\mathcal{D}_{a,b,c} = \emptyset$,

Theorem 2.2. *Let (a, b, c) be a triple of positive numbers with $a < b < c$, and let $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ be the Gabor system in (1.1) generated by the characteristic function on the interval $[0, c)$. Set*

$$c_0 = c - \lfloor c/b \rfloor b.$$

(V) ([25]) If $c_0 \geq a$ and $c_0 \leq b - a$, then $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame.

(VI) If $c_0 \geq a$ and $c_0 > b - a$, then $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame if and only if $a/b = p/q$ for some coprime integers, and either

- 1) $c_0 > b - \gcd([c/b] + 1, p)b/q$ and $\gcd([c/b] + 1, p) \neq [c/b] + 1$, or

- 2) $c_0 > b - \gcd(\lfloor c/b \rfloor + 1, p)b/q + b/q$ and $\gcd(\lfloor c/b \rfloor + 1, p) = \lfloor c/b \rfloor + 1$.
- (VII) If $c_0 < a$ and $c_0 \leq b - a$, then $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame if and only if either
- 3) $c_0 = 0$; or
- 4) $a/b = p/q$ for some coprime integers p and q , $0 < c_0 < \gcd(\lfloor c/b \rfloor, p)b/q$ and $\gcd(\lfloor c/b \rfloor, p) \neq \lfloor c/b \rfloor$; or
- 5) $a/b = p/q$ for some coprime integers p and q , $0 < c_0 < \gcd(\lfloor c/b \rfloor, p)b/q - b/q$ and $\gcd(\lfloor c/b \rfloor, p) = \lfloor c/b \rfloor$.

The conclusions in Theorem 2.2 are illustrated in the green, yellow and purple regions of Figure 1.

By Theorems 2.1 and 2.2, we now classify those triples (a, b, c) satisfying $a < b < c$ and $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, which turns out to be very complicated. Our approach is not to solve the linear system (2.3) directly, and instead to find binary vector solutions $\mathbf{x} \in \mathcal{B}_b^0$ of a “similar” infinite-dimensional linear system

$$(2.8) \quad \mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{1}.$$

Denote by $\mathcal{S}_{a,b,c}$ the periodic set of all $t \in \mathbb{R}$ such that there exists a binary vector solution to the linear system (2.8),

$$(2.9) \quad \mathcal{S}_{a,b,c} := \{t \in \mathbb{R} \mid \mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{1} \text{ for some vector } \mathbf{x} \in \mathcal{B}_b^0\}.$$

The set $\mathcal{S}_{a,b,c}$ just introduced is a superset of $\mathcal{D}_{a,b,c}$ in (2.2), and the set $\mathcal{D}_{a,b,c}$ can be obtained from $\mathcal{S}_{a,b,c}$ by some set operations,

$$\begin{aligned} \mathcal{D}_{a,b,c} = & (\mathcal{S}_{a,b,c} \cap ([0, c_0 + a - b] + a\mathbb{Z}) \cap (\mathcal{S}_{a,b,c} - \lfloor c/b \rfloor b)) \\ & \cup (\mathcal{S}_{a,b,c} \cap (\cup_{\lambda \in [b, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}} (\mathcal{S}_{a,b,c} - \lambda))), \end{aligned}$$

see (3.27) and Theorem 4.2. Hence the classification of triples (a, b, c) of positive numbers such that the Gabor system $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame reduces further to characterizing

$$(2.10) \quad \text{i) } \mathcal{S}_{a,b,c} = \emptyset,$$

and

$$(2.11) \quad \text{ii) } \mathcal{S}_{a,b,c} \neq \emptyset \text{ but } \mathcal{D}_{a,b,c} = \emptyset.$$

An advantage of the above approach is that there is a *unique* solution $\mathbf{x}_t \in \mathcal{B}_b^0$ to the infinite-dimensional linear system $\mathbf{M}_{a,b,c}(t)\mathbf{x}_t = \mathbf{1}$ for $t \in \mathcal{S}_{a,b,c}$, while multiple binary vector solutions could exist for the linear system (2.3) for $t \in \mathcal{D}_{a,b,c}$, see Proposition 3.9. Denote by $\lambda_{a,b,c}(t)$ the smallest positive index in $a\mathbb{Z}$ such that $\mathbf{x}_t(\lambda_{a,b,c}(t)) = 1$. This yields the following one-to-one map on the set $\mathcal{S}_{a,b,c}$:

$$(2.12) \quad \mathcal{S}_{a,b,c} \ni t \longleftrightarrow t + \lambda_{a,b,c}(t) \in \mathcal{S}_{a,b,c},$$

because $\tau_{\lambda_{a,b,c}(t)}\mathbf{x}_t \in \mathcal{B}_b^0$ and

$$\mathbf{M}_{a,b,c}(t + \lambda_{a,b,c}(t))\tau_{\lambda_{a,b,c}(t)}\mathbf{x}_t = \mathbf{M}_{a,b,c}(t)\mathbf{x}_t = \mathbf{1},$$

where $\tau_{\lambda'}\mathbf{z} := (\mathbf{z}(\lambda + \lambda'))_{\lambda \in b\mathbb{Z}}$ for $\mathbf{z} := (\mathbf{z}(\lambda))_{\lambda \in b\mathbb{Z}}$. Further inspection shows that the maps $t \rightarrow t + \lambda_{a,b,c}(t)$ and $t + \lambda_{a,b,c}(t) \rightarrow t$ on $\mathcal{S}_{a,b,c}$ can be extended to piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ on the real line \mathbb{R} . Here for a given triple (a, b, c) of positive numbers satisfying $a < b < c$ and $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, we define piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ on the real line \mathbb{R} by

$$(2.13) \quad R_{a,b,c}(t) := \begin{cases} t + \lfloor c/b \rfloor b + b & \text{if } t \in [0, c_0 + a - b) + a\mathbb{Z} \\ t & \text{if } t \in [c_0 + a - b, c_0) + a\mathbb{Z} \\ t + \lfloor c/b \rfloor b & \text{if } t \in [c_0, a) + a\mathbb{Z}, \end{cases}$$

and

$$(2.14) \quad \tilde{R}_{a,b,c}(t) := \begin{cases} t - \lfloor c/b \rfloor b & \text{if } t \in [c - a, c - c_0) + a\mathbb{Z} \\ t & \text{if } t \in [c - c_0, c + b - c_0 - a) + a\mathbb{Z} \\ t - \lfloor c/b \rfloor b - b & \text{if } t \in [c + b - c_0 - a, c) + a\mathbb{Z}. \end{cases}$$

Our extremely important observation is that $\mathcal{S}_{a,b,c}$ is the *maximal invariant set* under the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$,

$$R_{a,b,c}\mathcal{S}_{a,b,c} \subset \mathcal{S}_{a,b,c} \quad \text{and} \quad \tilde{R}_{a,b,c}\mathcal{S}_{a,b,c} \subset \mathcal{S}_{a,b,c},$$

that has empty intersection with their black holes $[c_0 + a - b, c_0) + a\mathbb{Z}$ and $[c - c_0, c - c_0 + b - a) + a\mathbb{Z}$, see Theorem 4.1.

Applying the above maximal invariant set property for $\mathcal{S}_{a,b,c}$, we take another step forward in the direction to solve the *abc*-problem for Gabor systems.

Theorem 2.3. *Let (a, b, c) be a triple of positive numbers with $a < b < c$ and $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, and let $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ be the Gabor system in (1.1) generated by the characteristic function on the interval $[0, c)$. Set*

$$c_1 = \lfloor c/b \rfloor b - \lfloor (\lfloor c/b \rfloor b / a) \rfloor a.$$

Then the following statements hold.

- (VIII) ([19, 25]) *If $\lfloor c/b \rfloor = 1$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame.*
- (IX) *If $\lfloor c/b \rfloor \geq 2$ and $c_1 > 2a - b$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame.*
- (X) *If $\lfloor c/b \rfloor \geq 2$ and $c_1 = 2a - b$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if $a/b = p/q$ for some coprime integers p and q , $c_0 \leq b - a + b/q$ and $\lfloor c/b \rfloor + 1 = p$.*

- (XI) If $\lfloor c/b \rfloor \geq 2$ and $c_1 = 0$, then $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if $a/b = p/q$ for some coprime integers p and q , $c_0 \geq a - b/q$ and $\lfloor c/b \rfloor = p$.

The conclusions in Theorem 2.3 are illustrated in the blue and dark blue regions of Figure 1.

By Theorems 2.1, 2.2 and 2.3, it remains to classify all triples (a, b, c) of positive numbers satisfying $a < b < c$, $\lfloor c/b \rfloor \geq 2$, $b - a < c_0 := c - \lfloor c/b \rfloor b < a$ and $0 < c_1 := \lfloor c/b \rfloor b - \lfloor (\lfloor c/b \rfloor b/a) \rfloor a < 2b - a$. For that purpose, we need explicit construction of the maximal invariant set $\mathcal{S}_{a,b,c}$ if it is nonempty. We observe that Hutchinson's remarkable construction [22] does not apply as piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ are *non-contractive*. On the other hand, from its maximal invariance, we obtain that for the triple $(a, b, c) = (\pi/4, 1, 23 - 11\pi/2)$ with irrational time-frequency-spacing ratio b/a , the maximal invariant set

$$\mathcal{S}_{a,b,c} = \left[18 - \frac{23\pi}{4}, 11 - \frac{7\pi}{2}\right) \cup \left[12 - \frac{15\pi}{4}, 5 - \frac{3\pi}{2}\right) \cup \left[6 - \frac{7\pi}{4}, 17 - \frac{21\pi}{4}\right) + \frac{\pi}{4}\mathbb{Z}$$

has its complement consisting of three holes of size $1 - \pi/4$ on one period, and that for the triple $(a, b, c) = (13/17, 1, 77/17)$ with rational time-frequency-spacing ratio b/a ,

$$\mathcal{S}_{a,b,c} = \left[\frac{2}{17}, \frac{3}{17}\right) \cup \left[\frac{9}{17}, \frac{10}{17}\right) \cup \left[\frac{12}{17}, \frac{13}{17}\right) + \frac{13}{17}\mathbb{Z}$$

is composed of three intervals of same length $1/17$ on the period $[0, 13/17)$, see Examples 5.1 and 5.2. A breakthrough of this paper is to show that if $\mathcal{S}_{a,b,c} \neq \emptyset$ then the black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$ of the transformation $R_{a,b,c}$ attracts the black hole $[c - c_0, c - c_0 + b - a) + a\mathbb{Z}$ of the other transformation $\tilde{R}_{a,b,c}$ when applying $R_{a,b,c}$ *finitely many times*, i.e.,

$$(R_{a,b,c})^L([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) = [c_0 + a - b, c_0) + a\mathbb{Z}$$

for some nonnegative integer L , and hence

$$(2.15) \quad \mathcal{S}_{a,b,c} = \mathbb{R} \setminus \left(\bigcup_{n=0}^L (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \right),$$

see Theorems 5.4 and 5.5, and Examples 5.1, 5.2 and 5.3. The above construction of the maximal invariant set $\mathcal{S}_{a,b,c}$ leads to the following characterization of (2.11): $\mathcal{S}_{a,b,c} \neq \emptyset$ but $\mathcal{D}_{a,b,c} = \emptyset$ if and only if $(\lfloor b/c \rfloor + 1)|\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b]| + \lfloor c/b \rfloor |\mathcal{S}_{a,b,c} \cap [c_0, a]| = a$, see Theorem 5.7.

Now it remains to discuss most challenging characterization of (2.10): classifying all triples (a, b, c) such that $\mathcal{S}_{a,b,c} \neq \emptyset$. By (2.15), the maximal invariant set $\mathcal{S}_{a,b,c}$ has its complement composed by finitely many holes on a period. So we may squeeze out those holes on the line and

then reconnect their endpoints. The above holes-removal surgery could be described by the map

$$(2.16) \quad Y_{a,b,c}(t) := \text{sgn}(t)|[\min(0, t), \max(0, t)) \cap \mathcal{S}_{a,b,c}|$$

on the line in the sense that it is an isomorphism from the set $\mathcal{S}_{a,b,c}$ to the *line with marks* (image of the holes). In Figure 2 below, we illustrate the performance of the holes-removal surgery via

$$a\mathbb{T} \ni a \exp(2\pi i t/a) \longmapsto Y_{a,b,c}(a) \exp(-2\pi i Y_{a,b,c}(t)/Y_{a,b,c}(a)) \in Y_{a,b,c}(a)\mathbb{T},$$

where $(\frac{\pi}{4}, 1, 23 - \frac{11\pi}{2})$, $(\frac{6}{7}, 1, \frac{23}{7})$, $(\frac{13}{17}, 1, \frac{77}{17})$ and $(\frac{13}{17}, 1, \frac{75}{17})$ are used as triples (a, b, c) in the four subfigures respectively, c.f. Examples 5.1, 5.2 and 5.3. In that figure, the blue arcs in the big circle are contained in $a \exp(2\pi i \mathcal{S}_{a,b,c}/a)$, the red dashed arcs in the big circle belong to $a \exp(2\pi i (\mathbb{R} \setminus \mathcal{S}_{a,b,c})/a)$, and the circled marks in the small circle are $Y_{a,b,c}(a) \exp(2\pi i K_{a,b,c}/Y_{a,b,c}(a))$, where $K_{a,b,c}$ is the set of all marks on the line.

Having the maximal invariant set $\mathcal{S}_{a,b,c}$ constructed in (2.15) and holes-removal surgery described by the map in (2.16), we next consider dynamic system of piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$. We observe that those piecewise linear transforms are not *measure-preserving* on the real line, but their restrictions on the maximal invariant set $\mathcal{S}_{a,b,c}$ are measure-preserving and one is the inverse of the other. More importantly, we show that there is a rotation $S(\theta_{a,b,c})$ of the circle $\mathbb{R}/(Y_{a,b,c}(a)\mathbb{Z})$,

$$(2.17) \quad S(\theta_{a,b,c})(z + Y_{a,b,c}(a)\mathbb{Z}) = \theta_{a,b,c} + z + Y_{a,b,c}(a)\mathbb{Z}, \quad z \in \mathbb{R}/(Y_{a,b,c}(a)\mathbb{Z}),$$

such that the following diagram commutes,

$$(2.18) \quad \begin{array}{ccc} \mathcal{S}_{a,b,c} & \xrightarrow{R_{a,b,c}} & \mathcal{S}_{a,b,c} \\ Y_{a,b,c} \downarrow & & \downarrow Y_{a,b,c} \\ \mathbb{R}/(Y_{a,b,c}(a)\mathbb{Z}) & \xrightarrow{S(\theta_{a,b,c})} & \mathbb{R}/(Y_{a,b,c}(a)\mathbb{Z}) \end{array}$$

see Theorem 5.8. In other words, the restriction of piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ on the maximal invariant set $\mathcal{S}_{a,b,c}$ can be thought as *rotations on the circle*, see Appendix C for the non-ergodicity of the piecewise linear transformation $R_{a,b,c}$ on the real line.

Having “rotation” property of the piecewise linear transformation $R_{a,b,c}$, we are almost ready to find all triples (a, b, c) such that $\mathcal{S}_{a,b,c} \neq \emptyset$. We observe that the hole-removal surgery is *reversible*, that is, the maximal invariant set $\mathcal{S}_{a,b,c}$ can be obtained from the real line by putting

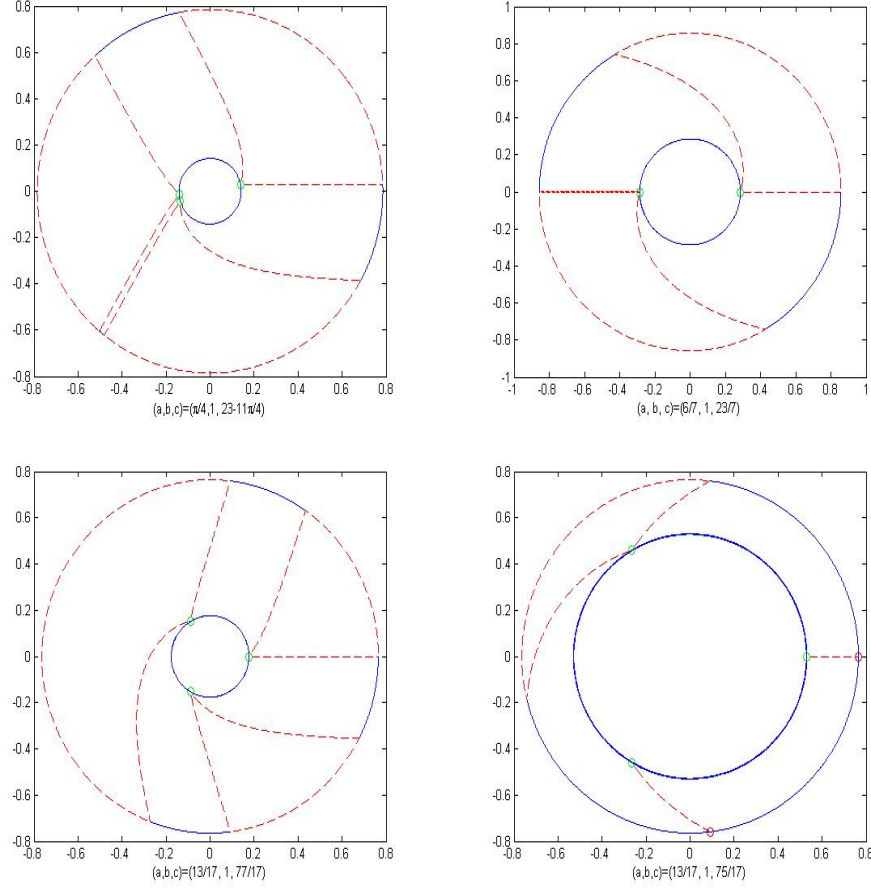


FIGURE 2. Holes-removal surgery for the maximal invariant set $\mathcal{S}_{a,b,c}$, see Examples 5.1, 5.2 and 5.3.

marks at appropriate positions and then inserting holes of appropriate sizes at marked positions. But that augmentation operation is much more delicate and complicated than the hole-removal surgery.

For the irrational time-frequency lattice case (i.e., $a/b \notin \mathbb{Q}$), holes to be inserted should have the same size $b-a$ (c.f. the upper-left subfigure of Figure 2) and the location of marks could be parameterized by the numbers of holes contained in the intervals $[0, c_0 + a - b)$ and $[c_0, a)$ respectively, see Theorem 6.1. This leads to a parametric characterization of the statement $\mathcal{S}_{a,b,c} \neq \emptyset$, and the following classification of triples (a, b, c) the irrational time-frequency lattice case.

Theorem 2.4. *Let (a, b, c) be a triple of positive numbers such that $a < b < c, b - a < c_0 := c - \lfloor c/b \rfloor b < a, \lfloor c/b \rfloor \geq 2, 0 < c_1 :=$*

$\lfloor c/b \rfloor b - \lfloor (\lfloor c/b \rfloor b/a) \rfloor a < 2a - b$. Let $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ be the Gabor system in (1.1) generated by the characteristic function on the interval $[0, c)$.

(XII) If $a/b \notin \mathbb{Q}$, then $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame if and only if there exist nonnegative integers d_1 and d_2 such that (a) $a \neq c - (d_1 + 1)(\lfloor c/b \rfloor + 1)(b - a) - (d_2 + 1)\lfloor c/b \rfloor(b - a) \in a\mathbb{Z}$; (b) $\lfloor c/b \rfloor b + (d_1 + 1)(b - a) < c < \lfloor c/b \rfloor b + b - (d_2 + 1)(b - a)$; and (c) $\#E_{a,b,c} = d_1$, where $m = ((d_1 + d_2 + 1)c_1 - c_0 + (d_1 + 1)(b - a))/a$ and

$$\begin{aligned} E_{a,b,c} &= \{n \in [1, d_1 + d_2 + 1] \mid n(c_1 - m(b - a)) \\ (2.19) \quad &\in [0, c_0 - (d_1 + 1)(b - a)) + (a - (d_1 + d_2 + 1)(b - a))\mathbb{Z}\}. \end{aligned}$$

The conclusion of Theorem 2.4 is illustrated in the white region of Figure 1. In the above theorem, we insert d_1 and d_2 holes contained in the intervals $[0, c_0 + a - b)$ and $[c_0, a)$ respectively, and put marks at $\cup_{n=1}^{d_1+d_2+1} (n(c_1 - m(b - a)) + (a - (d_1 + d_2 + 1)(b - a))\mathbb{Z})$.

For the rational time-frequency lattice case (i.e., $a/b \in \mathbb{Q}$), we write $a/b = p/q$ for some coprime integers p and q . Recall that for $c \notin b\mathbb{Z}/q$, $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if both $\mathcal{G}(\chi_{[0, \lfloor qc/b \rfloor b/q]}, a\mathbb{Z} \times \mathbb{Z}/b)$ and $\mathcal{G}(\chi_{[0, \lfloor qc/b+1 \rfloor b/q]}, a\mathbb{Z} \times \mathbb{Z}/b)$ are Gabor frames [25, Section 3.3.6.1]. Then it suffices to consider $a/b = p/q$ and $c/b \in \mathbb{Z}/q$ for some coprime integers p and q . In that case, we show that marks on the line are equally spaced and gaps to be inserted have two different sizes, which could be parameterized by the numbers of gaps of large and small sizes contained in intervals $[0, c_0 + a - b)$ and $[c_0, a)$, see Theorem 7.1 and c.f. Figure 2. Applying the characterization for $\mathcal{S}_{a,b,c} \neq \emptyset$ in Theorem 7.1, we reach the last step to solve the *abc*-problem for Gabor systems.

Theorem 2.5. *Let (a, b, c) be a triple of positive numbers such that $a < b < c$, $b - a < c_0 < a$, $\lfloor c/b \rfloor \geq 2$ and $0 < c_1 < 2a - b$, where $c_0 = c - \lfloor c/b \rfloor b$ and $c_1 = \lfloor c/b \rfloor b - \lfloor (\lfloor c/b \rfloor b/a) \rfloor a$. Let $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ be the Gabor system in (1.1) generated by the characteristic function on the interval $[0, c)$.*

(XIII) *If $a/b = p/q$ for some coprime integers p and q , and $c \in b\mathbb{Z}/q$, then $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame if and only if the triple (a, b, c) satisfies one of the following three conditions:*

- 6) $c_0 < \gcd(a, c_1)$ and $\lfloor c/b \rfloor(\gcd(a, c_1) - c_0) \neq \gcd(a, c_1)$.
- 7) $b - c_0 < \gcd(a, c_1 + b)$ and $(\lfloor c/b \rfloor + 1)(\gcd(a, c_1 + b) + c_0 - b) \neq \gcd(a, c_1 + b)$.
- 8) *There exist nonnegative integers d_1, d_2, d_3, d_4 such that (a) $0 < a - (d_1 + d_2 + 1)(b - a) \in Nb\mathbb{Z}/q$; (b) $Nc_1 + (d_1 +$*

$d_3 + 1)(b - a) \in a\mathbb{Z}$; (c) $(d_1 + d_2 + 1)(Nc_1 + (d_1 + d_3 + 1)(b - a)) - (d_1 + d_3 + 1)a \in Na\mathbb{Z}$; (d) $\gcd(Nc_1 + (d_1 + d_3 + 1)(b - a), Na) = a$; (e) $\#E_{a,b,c}^d = d_1$; (f) $c_0 = (d_1 + 1)(b - a) + (d_1 + d_3 + 1)(a - (d_1 + d_2 + 1)(b - a))/N + \delta$ for some $-\min(a - c_0, (a - (d_1 + d_2 + 1)(b - a))/N) < \delta < \min(c_0 + a - b, (a - (d_1 + d_2 + 1)(b - a))/N)$; and (g) $|\delta| + a/(N\lfloor c/b \rfloor + (d_1 + d_3 + 1)) \neq (a - (d_1 + d_2 + 1)(a - b))/N$, where $N := d_1 + d_2 + d_3 + d_4 + 2$ and $E_{a,b,c}^d$ is defined by

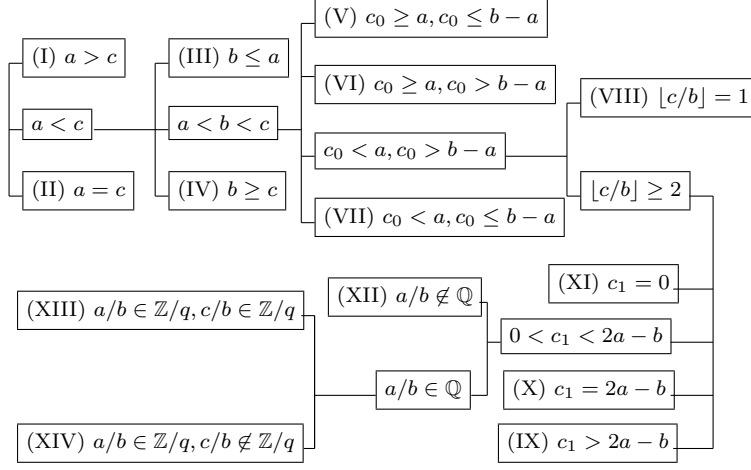
$$(2.20) \quad \begin{aligned} E_{a,b,c}^d &= \{n \in [1, d_1 + d_2 + 1] \mid n(Nc_1 + (d_1 + d_3 + 1)(b - a)) \\ &\in (0, (d_1 + d_3 + 1)a) + Na\mathbb{Z}\}. \end{aligned}$$

(XIV) ([25]) If $a < b < c, b - a < c_0 < a, \lfloor c/b \rfloor \geq 2, 0 < c_1 < 2a - b, a/b = p/q$ for some coprime integers p and q , and $c \notin b\mathbb{Z}/q$, then $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if both $\mathcal{G}(\chi_{[0, \lfloor qc/b \rfloor b/q]}, a\mathbb{Z} \times \mathbb{Z}/b)$ and $\mathcal{G}(\chi_{[0, \lfloor qc/b+1 \rfloor b/q]}, a\mathbb{Z} \times \mathbb{Z}/b)$ are Gabor frames.

The conclusions in the above theorem is illustrated in the white region of Figure 1. In Case 6) of Conclusion (XIII) in Theorem 2.5, the set $K_{a,b,c}$ of marks is $(\gcd(a, c_1) - c_0)\mathbb{Z}$ and gaps inserted at marked positions have same length c_0 . In Case 7) of Conclusion (XIII) in Theorem 2.5, $K_{a,b,c} = (\gcd(a, c_1 + b) + c_0 - b)\mathbb{Z}$ and gaps inserted at marks in $K_{a,b,c}$ are of size $b - c_0$. In Case 8) of Conclusion (XIII) in Theorem 2.5, $K_{a,b,c} = h\mathbb{Z}, Y_{a,b,c}(a) = Nh$ and gaps inserted at marked positions $lmh + Nh\mathbb{Z}, 1 \leq l \leq N$, have size $|b - a| + |\delta|$ for $1 \leq l \leq d_1 + d_2 + 1$ and $|\delta|$ for $d_1 + d_2 + 2 \leq l \leq N$, where $N = d_1 + d_2 + d_3 + d_4 + 2, h = (a - (d_1 + d_2 + 1)(b - a))/N - |\delta|, m = (Nc_1 + (d_1 + d_3 + 1)(b - a))/a$ and $\delta = c_0 - (d_1 + 1)(b - a) - (d_1 + d_3 + 1)(a - (d_1 + d_2 + 1)(b - a))/N$.

Combining Theorems 2.1–2.5 gives a complete answer to the abc -problem for Gabor systems. The classification diagram of triples (a, b, c)

in Theorems 2.1–2.5 is presented below:



From Classifications (V)–(IX) and (XII) in Theorems 2.2–2.4, it confirms a conjecture in [25, Section 3.3.5]: *If $a < b < c$, $a/b \notin \mathbb{Q}$ and $c \notin a\mathbb{Q} + b\mathbb{Q}$, then $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame for L^2 .* This, together with Classification (IV) in Theorem 2.1 and the shift-invariance, implies that the range of density parameters a, b such that $\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame is a dense subset of the open region $\mathcal{U}_c := \{(a, b) : 0 < a < \max(b, c)\}$, where c is the length of the interval I .

3. GABOR FRAMES AND TRIVIAL NULL SPACES OF INFINITE MATRICES

In this section, we prove Theorem 2.2.

To prove Theorem 2.2, we start from recalling some algebraic properties for infinite matrices in (2.1):

$$(3.1) \quad \mathbf{M}_{a,b,c}(t - \lambda')\mathbf{z} = \mathbf{M}_{a,b,c}(t)\tau_{\lambda'}\mathbf{z} \quad \text{for all } \lambda' \in b\mathbb{Z}$$

and

$$(3.2) \quad \mathbf{M}_{a,b,c}(t - \mu')\mathbf{z} = ((\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu + \mu'))_{\mu \in a\mathbb{Z}} \quad \text{for all } \mu' \in a\mathbb{Z},$$

where $t \in \mathbb{R}$, $\mathbf{z} := (\mathbf{z}(\lambda))_{\lambda \in b\mathbb{Z}}$, and $\tau_{\lambda'}\mathbf{z} := (\mathbf{z}(\lambda + \lambda'))_{\lambda \in b\mathbb{Z}}$. By the shift property (3.2) for infinite matrices $\mathbf{M}_{a,b,c}(t)$, $t \in \mathbb{R}$, the sets $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$ in (2.2) and (2.9) respectively are periodic sets with period a ,

$$(3.3) \quad \mathcal{D}_{a,b,c} = \mathcal{D}_{a,b,c} + a\mathbb{Z} \quad \text{and} \quad \mathcal{S}_{a,b,c} = \mathcal{S}_{a,b,c} + a\mathbb{Z}.$$

Next we recall the equivalence between frame property for the Gabor system $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ and uniform stability of infinite matrices $\mathbf{M}_{a,b,c}(t)$, $t \in \mathbb{R}$, in (2.1), which was established by Ron and Shen.

Lemma 3.1. ([32]) *Let (a, b, c) be a triple of positive numbers satisfying $\max(a, b) < c$, $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ be the Gabor system in (1.1), and let $\mathbf{M}_{a,b,c}(t), t \in \mathbb{R}$, be the infinite matrices in (2.1). Then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if there exist positive constants A and B such that*

$$(3.4) \quad A\|\mathbf{z}\|_2 \leq \|\mathbf{M}_{a,b,c}(t)\mathbf{z}\|_2 \leq B\|\mathbf{z}\|_2 \quad \text{for all } \mathbf{z} \in \ell^2 \text{ and } t \in \mathbb{R}.$$

Infinite matrices $\mathbf{M}_{a,b,c}(t), t \in \mathbb{R}$, in (2.1) have their rows containing $\lfloor c/b \rfloor + \{0, 1\}$ consecutive ones, and their rows are obtained by shifting one (or zero) unit of the previous row with possible reduction or expansion by one unit, c.f. (2.5) and (2.6). Due to the above special structures of infinite matrices in (2.1), we can reduce their uniform stability property (2.4) to the trivial intersection property between their null spaces $N(\mathbf{M}_{a,b,c}(t))$ and the set $\mathcal{B}_b - \mathcal{B}_b$ containing trinary vectors whose components take values in $\{-1, 0, 1\}$, and even further to the empty-set property for $\mathcal{D}_{a,b,c}$.

Theorem 3.2. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ be the Gabor system in (1.1), and let $\mathbf{M}_{a,b,c}(t), t \in \mathbb{R}$, be infinite matrices in (2.1). Then the following statements are equivalent to each other.*

- (i) *The Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame.*
- (ii) *For every $t \in \mathbb{R}$, only zero vector $\mathbf{0}$ is contained in the intersection between $\mathcal{B}_b - \mathcal{B}_b$ and the null space of the infinite matrix $\mathbf{M}_{a,b,c}(t)$; i.e.,*

$$N(\mathbf{M}_{a,b,c}(t)) \cap (\mathcal{B}_b - \mathcal{B}_b) = \{\mathbf{0}\} \quad \text{for every } t \in \mathbb{R}.$$

- (iii) $\mathcal{D}_{a,b,c} = \emptyset$.

In the next theorem, it is shown that it could further reduce the empty-set property for $\mathcal{D}_{a,b,c}$ in Theorem 3.2 to verifying whether it contains some particular points.

Theorem 3.3. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, the set $\mathcal{D}_{a,b,c}$ be as in (2.2), and define*

$$(3.5) \quad \mathcal{Z}_{a,b} := \begin{cases} \{0\} & \text{if } a/b \notin \mathbb{Q}, \\ \{0, b/q, \dots, b(p-1)/q\} & \text{if } a/b = p/q \text{ for some} \\ & \text{coprime integers } p \text{ and } q. \end{cases}$$

Then $\mathcal{D}_{a,b,c} = \emptyset$ if and only if $\mathcal{D}_{a,b,c} \cap (\mathcal{Z}_{a,b} \cup (c - \mathcal{Z}_{a,b})) = \emptyset$.

We remark that the implication (i) \implies (iii) in Theorem 3.2 has been applied implicitly in [19, 25] for their classification.

In the next three subsections, we prove Theorem 3.2, 2.2 and 3.3 respectively.

3.1. Trivial null spaces of infinite matrices. In this subsection, we prove Theorem 3.2. To do so, particularly the implication (iii) \implies (i), we need an equivalence between empty set property for the set $\mathcal{D}_{a,b,c}$ and uniform boundedness of maximal lengths of consecutive twos in the vector $\mathbf{M}_{a,b,c}(t)\mathbf{x}$ for any $t \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{B}_b$. Precisely, we define

$$(3.6) \quad Q_{a,b,c} := \sup_{t \in \mathbb{R}} Q_{a,b,c}(t) := \sup_{t \in \mathbb{R}} \left(\sup_{\mathbf{x} \in \mathcal{B}_b} Q_{a,b,c}(t, \mathbf{x}) \right),$$

where for any $t \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{B}_b$,

$$K(t, \mathbf{x}) := \{ \mu \in a\mathbb{Z} \mid \mathbf{M}_{a,b,c}(t)\mathbf{x}(\mu) = 2 \},$$

and

$$Q_{a,b,c}(t, \mathbf{x}) := \begin{cases} 0 & \text{if } K(t, \mathbf{x}) = \emptyset \\ \sup \{ n \in \mathbb{N} \mid [\mu, \mu + na) \cap a\mathbb{Z} \subset K(t, \mathbf{x}) \text{ for some } \mu \in a\mathbb{Z} \} & \text{otherwise} \end{cases}$$

is the maximal length of consecutive twos in the vector $\mathbf{M}_{a,b,c}(t)\mathbf{x}$. We show in Lemma 3.4 below that $\mathcal{D}_{a,b,c} = \emptyset$ if and only if the index $Q_{a,b,c}$ associated with infinite matrices $\mathbf{M}_{a,b,c}(t)$, $t \in \mathbb{R}$, is finite.

Lemma 3.4. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, and let $\mathbf{M}_{a,b,c}(t)$, $t \in \mathbb{R}$, be the infinite matrices in (2.1). Then $\mathcal{D}_{a,b,c} = \emptyset$ if and only if*

$$(3.7) \quad Q_{a,b,c} < +\infty.$$

The crucial and most technical step in the proof of Theorem 3.2 is to establish the following stability inequality:

$$(3.8) \quad \sum_{0 \leq \mu \leq aQ_{a,b,c} + 2a + b + c} |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu)| \geq \frac{b}{2c} |\mathbf{z}(0)|$$

for all $t \in [0, b)$ and vectors $\mathbf{z} = (\mathbf{z}(\lambda))_{\lambda \in b\mathbb{Z}}$.

Lemma 3.5. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, and let $Q_{a,b,c}$ be as in (3.6). If $Q_{a,b,c} < +\infty$, then (3.8) holds for any $t \in [0, b)$ and $\mathbf{z} = (\mathbf{z}(\lambda))_{\lambda \in b\mathbb{Z}}$.*

Now we start the proof of Theorem 3.2 by assuming that Lemmas 3.4 and 3.5 hold.

Proof of Theorem 3.2. (i) \implies (ii): Suppose, on the contrary, that there exist $t_0 \in \mathbb{R}$, and a nonzero vector $\mathbf{z}^* = (\mathbf{z}^*(\lambda))_{\lambda \in b\mathbb{Z}}$ such that

$$(3.9) \quad \mathbf{M}_{a,b,c}(t_0)\mathbf{z}^* = 0, \text{ and } \mathbf{z}^*(\lambda) \in \{-1, 0, 1\} \text{ for all } \lambda \in b\mathbb{Z}.$$

By Lemma 3.1 and the assumption (i), $\mathbf{M}_{a,b,c}(t_0)$ has the ℓ^2 -stability; i.e.,

$$(3.10) \quad 0 < \inf_{\|\mathbf{z}\|_2=1} \|\mathbf{M}_{a,b,c}(t_0)\mathbf{z}\|_2 \leq \sup_{\|\mathbf{z}\|_2=1} \|\mathbf{M}_{a,b,c}(t_0)\mathbf{z}\|_2 < \infty.$$

This together with (3.9) implies that

$$(3.11) \quad \mathbf{z}^* \notin \ell^2(b\mathbb{Z}).$$

Set $\mathbf{z}_N^* := (\mathbf{z}^*(\lambda)\chi_{[-N,N]}(\lambda))_{\lambda \in b\mathbb{Z}}$, $N \geq 2$. Then we obtain from (2.1), (3.9) and (3.11) that

$$\begin{cases} \lim_{N \rightarrow \infty} \|\mathbf{z}_N^*\|_2 = \infty, \\ \|\mathbf{M}_{a,b,c}(t_0)\mathbf{z}_N^*\|_\infty \leq \|\mathbf{M}_{a,b,c}(t_0)\mathbf{1}\|_\infty \leq c/b + 1, \text{ and} \\ (\mathbf{M}_{a,b,c}(t_0)\mathbf{z}_N^*)(\mu) = 0 \quad \text{for all } \mu - t_0 \notin [N - c, N] \cup [-N - c, -N]. \end{cases}$$

Therefore $\lim_{N \rightarrow \infty} \|\mathbf{M}_{a,b,c}(t_0)\mathbf{z}_N^*\|_2 / \|\mathbf{z}_N^*\|_2 = 0$, which contradicts to the ℓ^2 -stability (3.10).

(ii) \implies (iii): Suppose, on the contrary, that there exist $t_0 \in \mathbb{R}$ and a vector $\mathbf{x} \in \mathcal{B}_b^0$ such that $\mathbf{M}_{a,b,c}(t_0)\mathbf{x} = \mathbf{2}$. Denote by K the support of the vector \mathbf{x} ; i.e., the set of all $\lambda \in b\mathbb{Z}$ with $\mathbf{x}(\lambda) = 1$. Then from $\mathbf{M}_{a,b,c}(t_0)\mathbf{x} = \mathbf{2}$ it follows that

$$(3.12) \quad \#(K \cap (-t_0 + \mu + [0, c))) = 2 \quad \text{for all } \mu \in a\mathbb{Z}.$$

Thus $\#(K) = +\infty$ and $K = \{\lambda_j : j \in \mathbb{Z}\}$ for some strictly increasing sequence $\{\lambda_j\}_{j=-\infty}^\infty$ in $b\mathbb{Z}$. For any $\mu \in a\mathbb{Z}$, we obtain from (3.12) that $K \cap (-t_0 + \mu + [0, c))$ is either $\{\lambda_{2j}, \lambda_{2j+1}\}$ or $\{\lambda_{2j-1}, \lambda_{2j}\}$ for some $j \in \mathbb{Z}$. Thus

$$\#(K_i \cap (-t_0 + \mu + [0, c))) = 1 \quad \text{for all } \mu \in a\mathbb{Z} \text{ and } i = 0, 1,$$

where $K_i = \{\lambda_{i+2j} : j \in \mathbb{Z}\}$, $i = 0, 1$. Define $\mathbf{x}_i := (\mathbf{x}_i(\lambda))_{\lambda \in a\mathbb{Z}}$, $i = 0, 1$, by $\mathbf{x}_i(\lambda) = 1$ if $\lambda \in K_i$ and $\mathbf{x}_i(\lambda) = 0$ otherwise. Then \mathbf{x}_i , $i = 0, 1$, have one and only one of them in \mathcal{B}_b^0 while the other one in $\mathcal{B}_b \setminus \mathcal{B}_b^0$, and they satisfy

$$(3.13) \quad \mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1, \quad \mathbf{x}_0 \neq \mathbf{x}_1 \quad \text{and} \quad \mathbf{M}_{a,b,c}(t_0)\mathbf{x}_0 = \mathbf{M}_{a,b,c}(t_0)\mathbf{x}_1 = \mathbf{1}.$$

Thus $\mathbf{y} = \mathbf{x}_0 - \mathbf{x}_1$ is a nonzero vector in $\mathcal{B}_b - \mathcal{B}_b$ contained in the null space of the infinite matrix $\mathbf{M}_{a,b,c}(t_0)$, which contradicts to the assumption (ii).

(iii) \implies (i): Let $Q_{a,b,c}$ be as in (3.6). Then $Q_{a,b,c} < \infty$ by Lemma 3.4. For any $f \in L^2$,

$$\begin{aligned}
& \left(Q_{a,b,c} + \frac{3a+b+c}{a} \right)^2 \sum_{\phi \in \mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)} |\langle f, \phi \rangle|^2 \\
& \geq \left(Q_{a,b,c} + \frac{3a+b+c}{a} \right) \sum_{\mu \in a\mathbb{Z}} \sum_{0 \leq \mu' \leq aQ_{a,b,c} + 2a+b+c} \sum_{n \in \mathbb{Z}} \\
& \quad \left| \int_0^b \left(\sum_{\lambda \in b\mathbb{Z}} \chi_{[0,c]}(t - \mu' + \lambda) f(t + \mu + \lambda) \right) e^{-2\pi i n t/b} dt \right|^2 \\
& = b \sum_{\mu \in a\mathbb{Z}} \int_0^b \left(\left(Q_{a,b,c} + \frac{3a+b+c}{a} \right) \sum_{0 \leq \mu' \leq aQ_{a,b,c} + 2a+b+c} \left| \sum_{\lambda \in b\mathbb{Z}} \chi_{[0,c]}(t - \mu' + \lambda) f(t + \mu + \lambda) \right|^2 \right) dt \\
& \geq b \sum_{\mu \in a\mathbb{Z}} \int_0^b \left(\sum_{0 \leq \mu' \leq aQ_{a,b,c} + 2a+b+c} \left| \sum_{\lambda \in b\mathbb{Z}} \chi_{[0,c]}(t - \mu' + \lambda) f(t + \mu + \lambda) \right| \right)^2 dt \\
& \geq \frac{b^3}{4c^2} \sum_{\mu \in a\mathbb{Z}} \int_0^b |f(t + \mu)|^2 dt \geq \frac{b^3 \lfloor b/a \rfloor}{4c^2} \|f\|_2^2,
\end{aligned}$$

where the third inequality follows from Lemma 3.5, and the last inequality is true as $\sum_{\mu \in a\mathbb{Z}} \chi_{[0,b)}(t - \mu) \geq \lfloor b/a \rfloor$ for all $t \in \mathbb{R}$. Therefore $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame and the implication (iii) \implies (i) is established. \square

We finish this subsection by the proofs of Lemmas 3.4 and 3.5.

Proof of Lemma 3.4. (\Leftarrow) Suppose on the contrary that $\mathcal{D}_{a,b,c} \neq \emptyset$. Take $t_0 \in \mathcal{D}_{a,b,c}$ and a vector $\mathbf{x}_0 \in \mathcal{B}_b^0$ satisfying $\mathbf{M}_{a,b,c}(t_0)\mathbf{x}_0 = \mathbf{2}$. Then $Q_{a,b,c}(t, \mathbf{x}_0) = +\infty$, which implies that $Q_{a,b,c}(t) = +\infty$, a contradiction.

(\Rightarrow) Suppose, on the contrary, that $Q_{a,b,c} = +\infty$. Then for all $n \geq 1$ there exist $t_n \in \mathbb{R}$, $\mu_n \in a\mathbb{Z}$ and $\mathbf{x}_n \in \mathcal{B}_b$ such that $\mathbf{M}_{a,b,c}(t_n)\mathbf{x}_n(\mu) = 2$ for all $\mu_n \leq \mu \leq \mu_n + 2na$. By (3.1) and (3.2), without loss of generality, we may assume that $t_n \in [0, b)$ and

$$(3.14) \quad (\mathbf{M}_{a,b,c}(t_n)\mathbf{x}_n)(\mu) = 2 \quad \text{for all } \mu \in [-na, na] \cap a\mathbb{Z},$$

otherwise replacing t_n by the unique number $t'_n \in [0, b)$ satisfying $t_n - \mu_n - na - t'_n \in b\mathbb{Z}$ and \mathbf{x}_n by $\tau_{t'_n - t_n + \mu_n + na}\mathbf{x}_n$. Moreover, we can additionally assume that $\mathbf{x}_n := (\mathbf{x}_n(\mu))_{\mu \in b\mathbb{Z}} \in \mathcal{B}_b^0$, $n \geq 1$, satisfy

$$(3.15) \quad \mathbf{x}_{n'}(\lambda) = \mathbf{x}_n(\lambda) \quad \text{for all } \lambda \in [-nb, nb] \cap b\mathbb{Z} \text{ and } n' \geq n,$$

and

$$(3.16) \quad \{t_n\}_{n=1}^\infty \text{ is a monotone sequence,}$$

otherwise, replace $\{\mathbf{x}_n\}_{n=1}^\infty$ and $\{t_n\}_{n=1}^\infty$ by their subsequences satisfying (3.15) and (3.16). Denote by t_∞ and \mathbf{x}_∞ the limit of the sequence $\{t_n\}_{n=1}^\infty$ of numbers in $[0, b)$ and the sequence $\{\mathbf{x}_n\}_{n=1}^\infty$ of vectors in \mathcal{B}_b^0 respectively. Clearly $\mathbf{x}_\infty \in \mathcal{B}_b^0$.

If there exists n_0 such that $t_n = t_\infty$ for all $n \geq n_0$, then for any given $\mu \in a\mathbb{Z}$,

$$(\mathbf{M}_{a,b,c}(t_\infty)\mathbf{x}_\infty)(\mu) = (\mathbf{M}_{a,b,c}(t_n)\mathbf{x}_n)(\mu) = 2$$

for sufficiently large n by (3.15). Thus $\mathbf{M}_{a,b,c}(t_\infty)\mathbf{x}_\infty = \mathbf{2}$ and $t_\infty \in \mathcal{D}_{a,b,c}$, which contradicts to the assumption that $\mathcal{D}_{a,b,c} = \emptyset$.

If $\{t_n\}_{n=1}^\infty$ is a strictly decreasing sequence, then for any given $\lambda \in b\mathbb{Z}$ and $\mu \in a\mathbb{Z}$, $\chi_{[0,c]}(t_\infty - \mu + \lambda) = \chi_{[0,c]}(t_n - \mu + \lambda)$ for sufficiently large n . This together with (3.14) and (3.15) implies that

$$(\mathbf{M}_{a,b,c}(t_\infty)\mathbf{x}_\infty)(\mu) = \lim_{n \rightarrow \infty} (\mathbf{M}_{a,b,c}(t_n)\mathbf{x}_n)(\mu) = 2,$$

which contradicts to the assumption that $\mathcal{D}_{a,b,c} = \emptyset$.

If $\{t_n\}_{n=1}^\infty$ is a strictly increasing sequence, then for any given $\lambda \in a\mathbb{Z}$ and $\mu \in a\mathbb{Z}$, $\chi_{(0,c]}(t_\infty - \mu + \lambda) = \chi_{(0,c]}(t_n - \mu + \lambda)$ for sufficiently large n . This together with (3.14) and (3.15) yields that

$$\sum_{\lambda \in b\mathbb{Z}} \chi_{(0,c]}(t_\infty - \mu + \lambda) \mathbf{x}_\infty(\lambda) = 2 \quad \text{for all } \mu \in a\mathbb{Z},$$

or equivalently

$$\sum_{\lambda \in b\mathbb{Z}} \chi_{[0,c)}(c - t_\infty - \mu + \lambda) \mathbf{x}_\infty(-\lambda) = 2 \quad \text{for all } \mu \in a\mathbb{Z}.$$

Thus $c - t_\infty \in \mathcal{D}_{a,b,c}$, which is a contradiction. \square

Proof of Lemma 3.5. For $t \in [0, b)$, let $\lambda_0 = 0, \mu_0 = \lfloor t/a \rfloor a$ and let $\delta_0 \geq 0$ be the unique element in $[c + \mu_0 - t - b, c + \mu_0 - t) \cap b\mathbb{Z}$. If $\delta_0 = 0$, then (3.8) holds as $|(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_0)| = |\mathbf{z}(0)|$ and $\mu_0 \leq t \leq aQ_{a,b,c} + 2a + b + c$.

Now we prove (3.8) in the case that $\delta_0 \geq b$. To do so, we introduce families of triples $(\lambda_k^l, \mu_k^l, \delta_k^l) \in b\mathbb{Z} \times a\mathbb{Z} \times b\mathbb{Z}, 0 \leq k \leq M_l, 1 \leq l \leq \delta_0/b$, iteratively. For $i = 0, 1$, we let $\lambda_i^l = ilb, \mu_i^l = \lfloor (t + \lambda_i^l)/a \rfloor a$ and δ_i^l be the unique element in $[c + \mu_i^l - t - b, c + \mu_i^l - t) \cap b\mathbb{Z}$. Suppose that we have defined the triple $(\lambda_k^l, \mu_k^l, \delta_k^l)$, we set $M_l = k$ if $\delta_k^l \geq c + \mu_k^l - t + a - b$, and otherwise we define $(\lambda_{k+2}^l, \mu_{k+2}^l, \delta_{k+2}^l)$ by $\lambda_{k+2}^l = \delta_k^l + b, \mu_{k+2}^l = \lfloor (t + \lambda_{k+2}^l)/a \rfloor a$ and $\delta_{k+2}^l \in [c + \mu_{k+2}^l - t - b, c + \mu_{k+2}^l - t) \cap b\mathbb{Z}$. We remark that $\delta_k^l \geq c + \mu_k^l - t + a - b$ if the $(\mu_k^l + 1)$ -th row of the matrix $\mathbf{M}_{a,b,c}(t)$ is obtained by shift one unit to the left of the μ_k^l -th row with

reduction by one unit, c.f. the fourth and ninth rows in (2.5). Here is the visual interpretation of indices λ_k^l and μ_k^l , $1 \leq k \leq M_l$, in the matrix $\mathbf{M}_{a,b,c}(t)$:

$$\begin{array}{l} \mu_0 \rightarrow \\ \mu_1 \rightarrow \\ \mu_2 \rightarrow \\ \vdots \\ \mu_{M_l} \rightarrow \end{array} \left(\begin{array}{cccccccc} \lambda_0^l & & \lambda_1^l & & \lambda_2^l & & \lambda_3^l & & \lambda_4^l & \cdots & \lambda_{M_l}^l \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & & & & \\ & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & & & \\ & & & & \ddots & & & & & & \\ & & & 1 & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ & & & & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 \\ & & & & & & \ddots & & & & & \\ & & & & & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ & & & & & & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ & & & & & & & & & & \ddots & & \\ & & & & & & & & & & & 1 & 1 & \cdots & 1 \\ & & & & & & & & & & & & 1 & \cdots & 1 \end{array} \right).$$

From the above construction, we see triples $(\lambda_k^l, \mu_k^l, \delta_k^l)$, $0 \leq k \leq M_l$, have the following properties:

$$(3.17) \quad \begin{cases} \lambda_k^l \in [\mu_k^l - t, \mu_k^l - t + a) & \text{if } 0 \leq k \leq M_l \\ \lambda_{k+2}^l \in [c + \mu_k^l - t, c + \mu_k^l - t + a) & \text{if } 0 \leq k \leq M_l - 2, \end{cases}$$

$$(3.18) \quad [\mu_{M_l}^l - t + c, \mu_{M_l}^l - t + c + a) \cap b\mathbb{Z} = \emptyset \quad \text{if } M_l < \infty,$$

and

$$(3.19) \quad \{\lambda_k^l\}_{k=0}^{M_l} \text{ and } \{\mu_k^l\}_{k=0}^{M_l} \text{ are strictly increasing sequences.}$$

Define $\mathbf{x}_l := (\mathbf{x}_l(\lambda))_{\lambda \in b\mathbb{Z}}$ by $\mathbf{x}_l(\lambda) = 1$ if $\lambda = \lambda_k^l$ for some $0 \leq k \leq M_l$, and $\mathbf{x}_l(\lambda) = 0$ otherwise. Then $\mathbf{x}_l \in \mathcal{B}_b^0$ by (3.19), and for $\mu_0^l \leq \mu \leq \mu_{M_l-1}^l$,

$$\begin{aligned} (\mathbf{M}_{a,b,c}(t)\mathbf{x}_l)(\mu) &= \left(\sum_{0 \leq k \leq M_l, k \text{ even}} + \sum_{0 \leq k \leq M_l, k \text{ odd}} \right) \chi_{[0,c)}(t - \mu + \lambda_k^l) \\ &= \chi_{[0, \mu_0^l]}(\mu) + \sum_{2 \leq k \leq M_l, k \text{ even}} \chi_{(\mu_{k-2}^l, \mu_k^l]}(\mu) \\ &\quad + \chi_{(t + \lambda_1^l - c, \mu_1^l]}(\mu) + \sum_{3 \leq k \leq M_l, k \text{ odd}} \chi_{(\mu_{k-2}^l, \mu_k^l]}(\mu) = 2, \end{aligned}$$

where the second equation follows from

$$[\mu_{k-2}^l + a, \mu_k^l] \subset (t + \lambda_k^l - c, t + \lambda_k^l] \subset (\mu_{k-2}^l, \mu_k^l + a), \quad 2 \leq k \leq M_l$$

which holds by (3.17). This leads to the following crucial estimate:

$$(3.20) \quad \mu_{M_l-1}^l - \mu_0^l \leq aQ_{a,b,c}.$$

By (3.17) and (3.19), we have that

$$(3.21) \quad \mu_{M_l} - \mu_{M_l-1} \leq \mu_{M_l} - \mu_{M_l-2} \leq \lambda_{M_l}^l + t - (\lambda_{M_l}^l - c + t - a) \leq a + c.$$

Combining (3.20) and (3.21) and recalling $\mu_0 \leq t < b$, we obtain

$$(3.22) \quad \mu_{M_l} \leq aQ_{a,b,c} + a + b + c.$$

By (3.22), $M_l < \infty$ for all $1 \leq l \leq \delta_0/b$. Now we establish (3.8) if M_{l_0} is an even integer for some $1 \leq l_0 \leq \delta_0/b$. In this case, applying (3.17) and (3.18) with $l = l_0$, we obtain that

$$(3.23) \quad \begin{aligned} & |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k}^{l_0})| + |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k}^{l_0} + a)| \\ & \geq |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k}^{l_0}) - (\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k}^{l_0} + a)| \\ & = |\mathbf{z}(\lambda_{2k+2}^{l_0}) - \mathbf{z}(\lambda_{2k}^{l_0})| \end{aligned}$$

for all integers $0 \leq k \leq M_{l_0}/2 - 1$, and

$$(3.24) \quad |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k}^{l_0})| + |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k}^{l_0} + a)| \geq |\mathbf{z}(\lambda_{M_{l_0}}^{l_0})|$$

for $k = M_{l_0}/2$. Combining (3.22), (3.23) and (3.24), we get that

$$\begin{aligned} & 2 \sum_{0 \leq \mu \leq aQ_{a,b,c} + 2a + b + c} |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu)| \\ & \geq \sum_{k=0}^{M_{l_0}/2} |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k}^{l_0})| + |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k}^{l_0} + a)| \\ & \geq \sum_{k=0}^{M_{l_0}/2-1} |\mathbf{z}(\lambda_{2k+2}^{l_0}) - \mathbf{z}(\lambda_{2k}^{l_0})| + |\mathbf{z}(\lambda_{M_{l_0}}^{l_0})| \geq |\mathbf{z}(\lambda_0^{l_0})| = |\mathbf{z}(0)|. \end{aligned}$$

Hence (3.8) follows in the case that M_{l_0} is even for some $1 \leq l_0 \leq \delta_0/b$.

Finally we prove (3.8) in the case that M_l , $1 \leq l \leq \delta_0/b$, are all odd integers. In this case, mimicking the argument used to establish (3.23) and (3.24), we obtain that

$$(3.25) \quad |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k+1}^l)| + |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k+1}^l + a)| \geq |\mathbf{z}(\lambda_{2k+3}^l) - \mathbf{z}(\lambda_{2k+1}^l)|$$

for all integers $0 \leq k \leq (M_l - 3)/2$, and

$$(3.26) \quad |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k+1}^l)| + |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k+1}^l + a)| \geq |\mathbf{z}(\lambda_{M_l}^l)|$$

for $k = (M_l - 1)/2$, where $1 \leq l \leq \delta_0/b$. Therefore

$$\begin{aligned}
& \frac{2\delta_0}{b} \sum_{0 \leq \mu \leq aQ_{a,b,c} + 2a + b + c} |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu)| \\
& \geq |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_0)| + \sum_{1 \leq l \leq \delta_0/b} \sum_{k=0}^{(M_l-1)/2} \left(|(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k+1}^l)| \right. \\
& \quad \left. + |(\mathbf{M}_{a,b,c}(t)\mathbf{z})(\mu_{2k+1}^l + a)| \right) \\
& \geq \left| \sum_{0 \leq \lambda \leq \delta_0} \mathbf{z}(\lambda) \right| + \sum_{1 \leq l \leq \delta_0/b} |\mathbf{z}(\lambda_1^l)| \\
& = \left| \sum_{0 \leq \lambda \leq \delta_0} \mathbf{z}(\lambda) \right| + \sum_{1 \leq \lambda \leq \delta_0} |\mathbf{z}(\lambda)| \geq |\mathbf{z}(0)|
\end{aligned}$$

by (3.25) and (3.26). This together with $\delta_0 \leq c + \mu_0 - t \leq c$ proves (3.8) in the case that $M_l, 1 \leq l \leq \delta_0/b$, are all odd integers. This completes the proof of Lemma 3.5. \square

3.2. Proof of Theorem 2.2. In this subsection, we apply Theorem 3.2 to prove Theorem 2.2 by verifying whether $\mathcal{D}_{a,b,c}$ is an empty set or not. To do so, we notice that $\mathcal{S}_{a,b,c}$ is a supset of $\mathcal{D}_{a,b,c}$,

$$(3.27) \quad \mathcal{D}_{a,b,c} \subset \mathcal{S}_{a,b,c},$$

because any vector $\mathbf{x} \in \mathcal{B}_b^0$ satisfying $\mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{2}$ can be written as the sum of two vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{B}_b$ such that $\mathbf{M}_{a,b,c}(t)\mathbf{x}_1 = \mathbf{M}_{a,b,c}(t)\mathbf{x}_2 = \mathbf{1}$, c.f. (3.13) in the proof of Theorem 3.2. We also need some elementary properties for the supset $\mathcal{S}_{a,b,c}$, including empty intersection property (3.30) with the black holes.

Proposition 3.6. *Let (a, b, c) be a triple of positive numbers with $a < b < c$. Set $c_0 = c - \lfloor c/b \rfloor b$, $(c_0 + a - b)_+ = \max(c_0 + a - b, 0)$ and $c_0 \wedge a = \min(c_0, a)$. Define*

$$(3.28) \quad \lambda_{a,b,c}(t) = \begin{cases} \lfloor c/b \rfloor b + b & \text{if } t \in [0, (c_0 + a - b)_+] + a\mathbb{Z} \\ 0 & \text{if } t \in [(c_0 + a - b)_+, c_0 \wedge a) + a\mathbb{Z} \\ \lfloor c/b \rfloor b & \text{if } t \in [c_0 \wedge a, a) + a\mathbb{Z}, \end{cases}$$

and

$$(3.29) \quad \tilde{\lambda}_{a,b,c}(t) = \begin{cases} -\lfloor c/b \rfloor b - b & \text{if } t \in [c - (c_0 + a - b)_+, c) + a\mathbb{Z} \\ 0 & \text{if } t \in [c - c_0 \wedge a, c - (c_0 + a - b)_+) + a\mathbb{Z} \\ -\lfloor c/b \rfloor b & \text{if } t \in [c - a, c - c_0 \wedge a) + a\mathbb{Z}. \end{cases}$$

Then

$$(3.30) \quad \mathcal{S}_{a,b,c} \cap ([(c_0 + a - b)_+, c_0 \wedge a) \cup [c - c_0 \wedge a, c - (c_0 + a - b)_+] + a\mathbb{Z}) = \emptyset,$$

and

$$(3.31) \quad \mathbf{x}(\lambda) = \begin{cases} 0 & \text{if } \tilde{\lambda}_{a,b,c}(t) < \lambda < \lambda_{a,b,c}(t) \text{ and } \lambda \neq 0 \\ 1 & \text{if } \lambda = \lambda_{a,b,c}(t), 0, \tilde{\lambda}_{a,b,c}(t) \end{cases}$$

for any $t \in \mathcal{S}_{a,b,c}$ and $\mathbf{x} \in \mathcal{B}_b^0$ satisfying $\mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{1}$.

Now assuming that Proposition 3.6 holds, we start to prove Theorem 2.2. The main ideas behind our proof of Theorem 2.2 are as follows. To prove Conclusion (V) of Theorem 2.2, we use Proposition 3.6 to verify that $\mathcal{S}_{a,b,c} = \emptyset$, which together with (3.27) and Theorem 3.2 leads to the desired frame property for the Gabor system $\mathcal{G}(\chi_{[0,c]}a\mathbb{Z} \times \mathbb{Z}/b)$. The crucial step in the proof of Conclusion (VI) is that for any $t_0 \in \mathcal{D}_{a,b,c}$, the binary vector $\mathbf{x} \in \mathcal{B}_b^0$ satisfying $\mathcal{M}_{a,b,c}(t_0)\mathbf{x} = \mathbf{2}$ is supported on $(\lfloor c/b \rfloor + 1)b\mathbb{Z} \cup (\lambda_1 + (\lfloor c/b \rfloor + 1)b\mathbb{Z})$ for some $\lambda_1 \in [b, \lfloor c/b \rfloor b] \cap b\mathbb{Z}$, which implies that

$$\{t_0, t_0 + \lambda\} + (\lfloor c/b \rfloor + 1)b\mathbb{Z} + a\mathbb{Z} \subset \mathcal{D}_{a,b,c},$$

see (3.33) and (3.34). Define

$$(3.32) \quad \tilde{\mathbf{M}}_{a,b,c}(t) = (\chi_{(0,c]}(t - \mu + \lambda))_{\mu \in a\mathbb{Z}, \lambda \in b\mathbb{Z}}, \quad t \in \mathbb{R}.$$

We use similar argument to prove Conclusion (VII), except that for any $t_0 \in \tilde{\mathcal{D}}_{a,b,c} := c - \mathcal{D}_{a,b,c}$, the binary vector $\mathbf{x} \in \mathcal{B}_b^0$ satisfying $\tilde{\mathcal{M}}_{a,b,c}(t_0)\mathbf{x} = \mathbf{2}$ is supported on $\lfloor c/b \rfloor b\mathbb{Z} \cup (\lambda_1 + \lfloor c/b \rfloor b\mathbb{Z})$ for some $\lambda_1 \in [b, \lfloor c/b \rfloor b - 1] \cap b\mathbb{Z}$, see (3.39) and (3.40).

Proof of Theorem 2.2. (V): By (3.27) and Theorem 3.2, it suffices to prove $\mathcal{S}_{a,b,c} = \emptyset$, which follows from Proposition 3.6 as $[(c_0 + a - b)_+, c_0 \wedge a) + a\mathbb{Z} = [0, a) + a\mathbb{Z} = \mathbb{R}$ in this case. This conclusion was also established in [25, Section 3.3.3.2], we include the proof as the conclusion $\mathcal{S}_{a,b,c} = \emptyset$ will be used later to prove Theorem 3.3.

(VI): (\implies) By Theorem 3.2, $\mathcal{D}_{a,b,c} \neq \emptyset$. Then by (3.3) there exist $t_0 \in [0, a)$ and $\mathbf{x} \in \mathcal{B}_b^0$ such that $\mathbf{M}_{a,b,c}(t_0)\mathbf{x} = \mathbf{2}$. By (3.13), we can write $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1$ such that $\mathbf{x}_0 \in \mathcal{B}_b^0$, $\mathbf{x}_1 \in \mathcal{B}_b$ and $\mathbf{M}_{a,b,c}(t_0)\mathbf{x}_0 = \mathbf{M}_{a,b,c}(t_0)\mathbf{x}_1 = \mathbf{1}$. Let $\lambda_i, i = 1, 2, 3$, be the first three positive indices in $b\mathbb{Z}$ with $\mathbf{x}(\lambda_i) = 1, i = 1, 2, 3$. Then λ_2 is the first positive index in $b\mathbb{Z}$ with $\mathbf{x}_0(\lambda_2) = 1$ by (3.13). By Proposition 3.6, we obtain that $\lambda_2 = \lambda_3 - \lambda_1 = \lfloor c/b \rfloor b + b$. Further inspection also shows that $t_0 \in [0, c_0 + a - b)$, $t_0 + \lambda_1 \in [0, c_0 + a - b) + a\mathbb{Z}$, and the unique number $t_1 \in [0, a) \cap (t_0 + \lambda_2 + a\mathbb{Z})$ belongs to $\mathcal{D}_{a,b,c}$, where $0 \leq c_0 := c - \lfloor c/b \rfloor b < b$. Inductively for any $n \geq 1$, the unique number t_n in $[0, a) \cap (t_{n-1} + (\lfloor c/b \rfloor + 1)b + a\mathbb{Z})$

belongs to $\mathcal{D}_{a,b,c} \cap [0, c_0 + a - b)$ and $t_n + \lambda_1 \in \mathcal{D}_{a,b,c} \cap ([0, c_0 + a - b) + a\mathbb{Z})$. From the above construction, we see that $t_n - t_0 - n(\lfloor c/b \rfloor + 1)b \in a\mathbb{Z}$. Hence a/b is rational, as otherwise $\{t_n : n \geq 0\}$ is dense in $[0, a)$, which is a contradiction as $c_0 + a - b < a$ and $t_n + \lambda_1 \in [0, c_0 + a - b) + a\mathbb{Z}$ for all $n \geq 0$. Now we write $a/b = p/q$ for some coprime integers p and q , set $m = \gcd(\lfloor c/b \rfloor + 1, p)$, and let t'_0 be the unique number in $[0, mb/q)$ such that $t'_0 - t_0 \in mb\mathbb{Z}/q$. Then

$$\begin{aligned}
 \{t_n : n \geq 0\} + a\mathbb{Z} &= \{t_0 + n(\lfloor c/b \rfloor + 1)b : n \geq 0\} + a\mathbb{Z} \\
 &= t_0 + (\lfloor c/b \rfloor + 1)b\mathbb{Z} + a\mathbb{Z} = t_0 + mb\mathbb{Z}/q \\
 (3.33) \quad &= \{t'_0 + nmb/q : 0 \leq n \leq p/m - 1\} + a\mathbb{Z},
 \end{aligned}$$

where the first equality follows from the definition of t_n , the second one holds as $p(\lfloor c/b \rfloor + 1)b \in a\mathbb{Z}$, and the last one is true by $m = \gcd((\lfloor c/b \rfloor + 1)q, p)$. Similarly, we have that

$$(3.34) \quad \{t_n + \lambda_1 : n \geq 0\} + a\mathbb{Z} = \{t''_0 + nmb/q : 0 \leq n \leq p/m - 1\} + a\mathbb{Z},$$

where t''_0 is the unique number in $[0, mb/q)$ such that $t''_0 - t_0 - \lambda_1 \in mb\mathbb{Z}/q$. Hence from (3.33), (3.34) and the property that $t_n, t_n + \lambda_1 \in [0, c_0 + a - b) + a\mathbb{Z}, n \geq 1$, we obtain that $\{t'_0 + nmb/q : 0 \leq n \leq p/m - 1\} \subset [0, c_0 + a - b)$ and $\{t''_0 + nmb/q : 0 \leq n \leq p/m - 1\} \subset [0, c_0 + a - b)$. Observe that $t''_0 - t'_0 \in \lambda_1 + mb\mathbb{Z}/q \subset b\mathbb{Z}/q$, and that $t''_0 - t'_0 \neq 0$ if $m = \lfloor c/b \rfloor + 1$ as otherwise $(\lambda_1/b)q = (\lfloor c/b \rfloor + 1)r$ for some $r \in \mathbb{Z}$. This together with $1 = \gcd(q, p) \geq \gcd(q, m)$ implies that $\lambda_1 \in (\lfloor c/b \rfloor + 1)b\mathbb{Z}$, which is a contradiction as $0 < \lambda_1 < (\lfloor c/b \rfloor + 1)b$. Therefore $c_0 + a - b > (p/m - 1)mb/q$ if $m \neq \lfloor c/b \rfloor + 1$ and $c_0 + a - b > (p/m - 1)mb/q + b/q$ if $m = \lfloor c/b \rfloor + 1$. This completes the proof of the necessity.

(\Leftarrow) In the case that $\gcd(\lfloor c/b \rfloor + 1, p) \neq \lfloor c/b \rfloor + 1$, we define $\mathbf{x}(\lambda) = 1$ if $\lambda \in \{0, \gcd(\lfloor c/b \rfloor + 1, p)b\} + (\lfloor c/b \rfloor + 1)b\mathbb{Z}$ and $\mathbf{x}(\lambda) = 0$ otherwise. Then $\mathbf{x} \in \mathcal{B}_b^0$ and

$$\begin{aligned}
 \mathbf{M}_{a,b,c}(0)\mathbf{x}(\mu) &= \sum_{k \in \mathbb{Z}} \chi_{[0,c)}(k(\lfloor c/b \rfloor + 1)b - \mu) \\
 &\quad + \sum_{k \in \mathbb{Z}} \chi_{[0,c)}(k(\lfloor c/b \rfloor + 1)b + \gcd(\lfloor c/b \rfloor + 1, p)b - \mu) \\
 &= \sum_{i=1}^2 \sum_{k \in \mathbb{Z}} \chi_{[0,c)}(k(\lfloor c/b \rfloor + 1)b + s_i \gcd(\lfloor c/b \rfloor + 1, p)b/q) \\
 (3.35) \quad &= \sum_{i=1}^2 \chi_{[0,c)}(s_i \gcd(\lfloor c/b \rfloor + 1, p)b/q) = 2, \quad \mu \in a\mathbb{Z},
 \end{aligned}$$

where $0 \leq s_1, s_2 \leq q(\lfloor c/b \rfloor + 1)/\gcd(\lfloor c/b \rfloor + 1, p) - 1$ are so chosen that $-\mu - s_1 \gcd(\lfloor c/b \rfloor + 1, p)b/q \in (\lfloor c/b \rfloor + 1)b\mathbb{Z}$ and $-\mu + \gcd(\lfloor c/b \rfloor + 1, p)b/q \in (\lfloor c/b \rfloor + 1)b\mathbb{Z}$.

$1, p)b - s_2 \gcd(\lfloor c/b \rfloor + 1, p)b/q \in (\lfloor c/b \rfloor + 1)b\mathbb{Z}$. Therefore $\mathcal{D}_{a,b,c} \neq \emptyset$ and hence the corresponding Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame by Theorem 3.2.

In the case that $\gcd(\lfloor c/b \rfloor + 1, p) = \lfloor c/b \rfloor + 1$, we have that $p \geq 2$ and we define $\mathbf{x}(\lambda) = 1$ if $\lambda \in \{0, \lambda_1\} + (\lfloor c/b \rfloor + 1)b\mathbb{Z}$ and $\mathbf{x}(\lambda) = 0$ otherwise, where $\lambda_1 \in b\mathbb{Z}$ is chosen so that $b \leq \lambda_1 \leq (p-1)b$ and $q\lambda_1/b - 1 \in p\mathbb{Z}$. Then

$$\begin{aligned}
\mathbf{M}_{a,b,c}(0)\mathbf{x}(\mu) &= \sum_{k \in \mathbb{Z}} \chi_{[0,c)}(k(\lfloor c/b \rfloor + 1)b - \mu) \\
&\quad + \sum_{k \in \mathbb{Z}} \chi_{[0,c)}(k(\lfloor c/b \rfloor + 1)b + \lambda_1 - \mu) \\
&= \sum_{k \in \mathbb{Z}} \chi_{[0,c)}(k(\lfloor c/b \rfloor + 1)b + s_3(\lfloor c/b \rfloor + 1)b/q) \\
&\quad + \sum_{k \in \mathbb{Z}} \chi_{[0,c)}(k(\lfloor c/b \rfloor + 1)b + s_4(\lfloor c/b \rfloor + 1)b/q + b/q) \\
&= \chi_{[0,c)}(s_3(\lfloor c/b \rfloor + 1)b/q) \\
(3.36) \quad &\quad + \chi_{[0,c)}(s_4(\lfloor c/b \rfloor + 1)b/q + b/q) = 2
\end{aligned}$$

where $0 \leq s_3, s_4 \leq q-1$ are so chosen that $-\mu - s_3(\lfloor c/b \rfloor + 1)b/q \in (\lfloor c/b \rfloor + 1)b\mathbb{Z}$ and $-\mu + \lambda_1 - b/q - s_4(\lfloor c/b \rfloor + 1)b/q \in (\lfloor c/b \rfloor + 1)b\mathbb{Z}$, and the last equality follows from the assumption on c_0 . Therefore $\mathcal{D}_{a,b,c} \neq \emptyset$ in the case that $\gcd(\lfloor c/b \rfloor + 1, p) = \lfloor c/b \rfloor + 1$ and hence the Gabor system $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame by Theorem 3.2.

(VII): Let $\tilde{\mathbf{M}}_{a,b,c}(t), t \in \mathbb{R}$, be as in (3.32) and define

$$(3.37) \quad \tilde{\mathcal{D}}_{a,b,c} = c - \mathcal{D}_{a,b,c}.$$

Then

$$(3.38) \quad \tilde{\mathcal{D}}_{a,b,c} = \{t \in \mathbb{R} : \tilde{\mathbf{M}}_{a,b,c}(t)\mathbf{x} = \mathbf{2} \text{ for some } \mathbf{x} \in \mathcal{B}_b^0\}.$$

We will apply an argument similar to the one used in the proof of the conclusion (VI) essentially replacing $\mathcal{M}_{a,b,c}(t)$, $\mathcal{D}_{a,b,c}$ and $\lfloor c/b \rfloor + 1$ by $\tilde{\mathcal{M}}_{a,b,c}(t)$, $\tilde{\mathcal{D}}_{a,b,c}$ and $\lfloor c/b \rfloor$ respectively.

(\implies) Let $t_0 \in \tilde{\mathcal{D}}_{a,b,c}$ for some $t_0 \in (0, a]$. The existence of such a number t_0 follows from (3.1), (3.37) and Theorem 3.2. Then by (3.32) and (3.38), $\tilde{\mathbf{M}}_{a,b,c}(t_0)\mathbf{x} = \mathbf{2}$ for some $\mathbf{x} \in \mathcal{B}_b^0$. Following the argument to prove (3.13), we can write $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1$ for some $\mathbf{x}_0 \in \mathcal{B}_b^0$ and $\mathbf{x}_1 \in \mathcal{B}_b$ satisfying $\tilde{\mathbf{M}}_{a,b,c}(t_0)\mathbf{x}_0 = \tilde{\mathbf{M}}_{a,b,c}(t_0)\mathbf{x}_1 = \mathbf{1}$, and let $\lambda_i, i = 1, 2, 3$, be the first three positive indices in $b\mathbb{Z}$ with $\mathbf{x}(\lambda_i) = 1, i = 1, 2, 3$. Then λ_2 is the first positive index in $b\mathbb{Z}$ with $\mathbf{x}_0(\lambda_2) = 1$, and $\lambda_2 = \lambda_3 - \lambda_1 = \lfloor c/b \rfloor b$ by following the argument used in the proof of Proposition 3.6.

Furthermore $t_0 \in (c_0, a]$, $t_0 + \lambda_1 \in (c_0, a] + a\mathbb{Z}$, and the unique number $t_1 \in (0, a]$ with $t_1 - t_0 - \lambda_2 \in a\mathbb{Z}$ belongs to $\tilde{\mathcal{D}}_{a,b,c}$. Inductively for any $n \geq 1$, the unique number $t_n \in (0, a]$ satisfying $t_n - t_{n-1} - \lfloor c/b \rfloor b \in a\mathbb{Z}$ belongs to $\tilde{\mathcal{D}}_{a,b,c} \cap (c_0, a]$ and $t_n + \lambda_1 \in \tilde{\mathcal{D}}_{a,b,c} \cap ((c_0, a] + a\mathbb{Z})$. From the above construction, $t_n - t_0 - n\lfloor c/b \rfloor b \in a\mathbb{Z}$. Hence $a/b \in \mathbb{Q}$ if $c_0 \neq 0$, as otherwise $\{t_n : n \geq 0\}$ is dense in $(0, a]$, which is a contradiction. Now consider $c_0 \neq 0$ and write $a/b = p/q$ for some coprime integers p and q , set $m = \gcd(\lfloor c/b \rfloor, p)$. Let t'_0 and t''_0 be the unique number in $(0, mb/q]$ such that $t'_0 - t_0 \in mb\mathbb{Z}/q$ and $t''_0 - t_0 - \lambda_1 \in mb\mathbb{Z}/q$. Then

$$(3.39) \quad \{t_n : n \geq 0\} + a\mathbb{Z} = \{t'_0 + nmb/q : 0 \leq n \leq p/m - 1\} + a\mathbb{Z},$$

and

$$(3.40) \quad \{t_n + \lambda_1 : n \geq 0\} + a\mathbb{Z} = \{t''_0 + nmb/q : 0 \leq n \leq p/m - 1\} + a\mathbb{Z}.$$

Hence from (3.39), (3.40) and the property that $t_n, t_n + \lambda_1 \in (c_0, a] + a\mathbb{Z}$, $n \geq 1$, it follows that $\{t'_0 + nmb/q : 0 \leq n \leq p/m - 1\} \subset (c_0, a]$ and $\{t''_0 + nmb/q : 0 \leq n \leq p/m - 1\} \subset (c_0, a]$. Observe that $t''_0 - t'_0 \in \lambda_1 + mb\mathbb{Z}/q \subset b\mathbb{Z}/q$ and that $t''_0 - t'_0 \neq 0$ if $m = \lfloor c/b \rfloor$ as otherwise $\lambda_1 \in mb\mathbb{Z}$, which is a contradiction. Therefore $0 < c_0 < mb/q$ if $m \neq \lfloor c/b \rfloor$ and $0 < c_0 < mb/q - b/q$ if $m = \lfloor c/b \rfloor$. This completes the proof of the necessity.

(\Leftarrow) In the case that $c_0 = 0$, we define $\mathbf{x}(\lambda) = 1$ if $\lambda \in \{0, b\} + \lfloor c/b \rfloor b\mathbb{Z}$ and $\mathbf{x}(\lambda) = 0$ otherwise. Then $\mathbf{x} \in \mathcal{B}_b^0$ and

$$\mathbf{M}_{a,b,c}(0)\mathbf{x}(\mu) = \sum_{k \in \mathbb{Z}} \chi_{[0,c)}(-\mu + kc) + \chi_{[0,c)}(-\mu + b + kc) = 2$$

for all $\mu \in a\mathbb{Z}$. Then $\mathcal{D}_{a,b,c} \neq \emptyset$ and $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame by Theorem 3.2.

In the case that $a/b = p/q$ for some coprime integers p and q , $0 < c_0 < \gcd(\lfloor c/b \rfloor, p)b/q$ and $\gcd(\lfloor c/b \rfloor, p) \neq \lfloor c/b \rfloor$, we define $\mathbf{x}(\lambda) = 1$ if $\lambda \in \{0, \gcd(\lfloor c/b \rfloor, p)b\} + \lfloor c/b \rfloor b\mathbb{Z}$ and $\mathbf{x}(\lambda) = 0$ otherwise. Then $\mathbf{x} \in \mathcal{B}_b^0$ and

$$(3.41) \quad \begin{aligned} \tilde{\mathbf{M}}_{a,b,c}(0)\mathbf{x}(\mu) &= \sum_{k \in \mathbb{Z}} \chi_{(0,c]}(s_1 \gcd(\lfloor c/b \rfloor, p)b/q + k\lfloor c/b \rfloor b) \\ &+ \sum_{k \in \mathbb{Z}} \chi_{(0,c]}(s_2 \gcd(\lfloor c/b \rfloor, p)b/q + k\lfloor c/b \rfloor b) = 2, \end{aligned}$$

where $1 \leq s_1, s_2 \leq q\lfloor c/b \rfloor / \gcd(\lfloor c/b \rfloor, p)$ are so chosen that $-\mu - s_1 \gcd(\lfloor c/b \rfloor, p)b/q \in \lfloor c/b \rfloor b\mathbb{Z}$ and $-\mu + \gcd(\lfloor c/b \rfloor, p)b - s_2 \gcd(\lfloor c/b \rfloor, p)b/q \in \lfloor c/b \rfloor b\mathbb{Z}$. Hence $0 \in \tilde{\mathcal{D}}_{a,b,c}$ (or equivalently $c \in \mathcal{D}_{a,b,c}$ by (3.38)) and $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame by Theorem 3.2.

In the case that $a/b = p/q$ for some coprime integers p and q , $0 < c_0 < \gcd(\lfloor c/b \rfloor, p)b/q - b/q$ and $\gcd(\lfloor c/b \rfloor, p) = \lfloor c/b \rfloor$, we have that $\lfloor c/b \rfloor \geq 2$ and define $\mathbf{x}(\lambda) = 1$ if $\lambda \in \{0, \lambda_1\} + \lfloor c/b \rfloor b\mathbb{Z}$ and $\mathbf{x}(\lambda) = 0$ otherwise, where $b \leq \lambda_1 \leq (\lfloor c/b \rfloor - 1)b$ is so chosen that $q\lambda_1/b + 1 \in \lfloor c/b \rfloor \mathbb{Z}$. Then $\mathbf{x} \in \mathcal{B}_b^0$ and

$$(3.42) \quad \begin{aligned} \tilde{\mathbf{M}}_{a,b,c}(0)\mathbf{x}(\mu) &= \sum_{k \in \mathbb{Z}} \chi_{(0,c]}(s_3 \gcd(\lfloor c/b \rfloor, p)b/q + k\lfloor c/b \rfloor b) \\ &+ \sum_{k \in \mathbb{Z}} \chi_{(0,c]}(s_4 \gcd(\lfloor c/b \rfloor, p)b/q - b/q + k\lfloor c/b \rfloor b) = 2, \end{aligned}$$

where $1 \leq s_3, s_4 \leq q\lfloor c/b \rfloor / \gcd(\lfloor c/b \rfloor, p)$ are so chosen that $-\mu - s_3 \gcd(\lfloor c/b \rfloor, p)b/q \in \lfloor c/b \rfloor b\mathbb{Z}$ and $-\mu + \lambda_1 - b/q - s_4 \gcd(\lfloor c/b \rfloor, p)b/q \in \lfloor c/b \rfloor b\mathbb{Z}$. This proves that $0 \in \tilde{\mathcal{D}}_{a,b,c}$ and hence $c \in \mathcal{D}_{a,b,c}$ by (3.38). Thus $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame by Theorem 3.2. \square

We finish this subsection with the proof of Proposition 3.6.

Proof of Proposition 3.6. Let $t \in \mathcal{S}_{a,b,c}$ and $\mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{1}$ for some $\mathbf{x} \in \mathcal{B}_b^0$. By (3.2), we may assume that $t \in [0, a)$. Let $\lambda_1 \in b\mathbb{Z}$ be the smallest positive index such that $\mathbf{x}(\lambda_1) = 1$. Then $\lambda_1 \geq \lfloor c/b \rfloor b$ because

$$1 = \chi_{[0,c)}(t) \leq \chi_{[0,c)}(t) + \chi_{[0,c)}(t + \lambda_1) \leq \sum_{\lambda \in b\mathbb{Z}} \chi_{[0,c)}(t + \lambda)\mathbf{x}(\lambda) = 1,$$

and $\lambda_1 \leq \lfloor c/b \rfloor b + b$ since otherwise $\sum_{\lambda \in b\mathbb{Z}} \chi_{[0,c)}(t - a + \lambda)\mathbf{x}(\lambda) = 0$. Thus λ_1 is either $\lfloor c/b \rfloor b$ or $\lfloor c/b \rfloor b + b$. If $\lambda_1 = \lfloor c/b \rfloor b$, then $t \geq c_0$ (and hence $\min(c_0, a) \leq t < a$) by (3.43). If $\lambda_1 = \lfloor c/b \rfloor b + b$, then $t < c_0 + a - b$ as $1 = \sum_{\lambda \in b\mathbb{Z}} \chi_{[0,c)}(t - a + \lambda)\mathbf{x}(\lambda) = \chi_{[0,c)}(t - a + \lfloor c/b \rfloor b + b)$. This proves that $t \notin [(c_0 + a - b)_+, \min(c_0, a))$ and $\lambda_1 = \lambda_{a,b,c}(t)$. Hence

$$(3.44) \quad \mathcal{S}_{a,b,c} \cap ((c_0 + a - b)_+, c_0 \wedge a) + a\mathbb{Z} = \emptyset$$

and

$$(3.45) \quad \mathbf{x}(\lambda) = \begin{cases} 0 & \text{if } 0 < \lambda < \lambda_{a,b,c}(t), \\ 1 & \text{if } \lambda = \lambda_{a,b,c}(t). \end{cases}$$

For the above vector \mathbf{x} , we have that $\tilde{\mathbf{M}}_{a,b,c}(c - t)\tilde{\mathbf{x}} = \mathbf{1}$, where $\tilde{\mathbf{M}}_{a,b,c}(t)$ is given in (3.32) and $\tilde{\mathbf{x}} = (\mathbf{x}(-\lambda))_{\lambda \in b\mathbb{Z}} \in \mathcal{B}_b^0$. Then mimicking the above argument to establish (3.44) and (3.45), we obtain that

$$(3.46) \quad \mathcal{S}_{a,b,c} \cap ([c - c_0 \wedge a, c - (c_0 + a - b)_+) + a\mathbb{Z} = \emptyset,$$

and

$$(3.47) \quad \tilde{\mathbf{x}}(\lambda) = \begin{cases} 0 & \text{if } 0 < \lambda < -\tilde{\lambda}_{a,b,c}(t) \\ 1 & \text{if } \lambda = -\tilde{\lambda}_{a,b,c}(t). \end{cases}$$

Combining (3.44)–(3.47), we obtain (3.30) and (3.31). \square

3.3. Binary solutions of infinite-dimensional linear systems. In this subsection, we prove Theorem 3.3. To do so, we first recall some basic properties concerning black holes, ranges, invertibility of piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ on the real line.

Proposition 3.7. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$ and $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, and let $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ be infinite matrices in (2.13) and (2.14). Then*

$$(3.48) \quad \begin{cases} R_{a,b,c}(t) = t & \text{if } t \in [c_0 + a - b, c_0) + a\mathbb{Z} \\ \tilde{R}_{a,b,c}(t) = t & \text{if } t \in [c - c_0, c + b - c_0 - a) + a\mathbb{Z}, \end{cases}$$

$$(3.49)$$

$$\begin{cases} R_{a,b,c}(\mathbb{R} \setminus ([c_0 + a - b, c_0) + a\mathbb{Z})) = \mathbb{R} \setminus ([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) \\ \tilde{R}_{a,b,c}(\mathbb{R} \setminus ([c - c_0, c + b - c_0 - a) + a\mathbb{Z})) = \mathbb{R} \setminus ([c_0 + a - b, c_0) + a\mathbb{Z}), \end{cases}$$

and

$$(3.50) \quad \begin{cases} \tilde{R}_{a,b,c}(R_{a,b,c}(t)) = t & \text{for all } t \notin [c_0 + a - b, c_0) + a\mathbb{Z} \\ R_{a,b,c}(\tilde{R}_{a,b,c}(t)) = t & \text{for all } t \notin [c - c_0, c + b - c_0 - a) + a\mathbb{Z}. \end{cases}$$

The black hole property (3.48), the range property (3.49), and the invertibility property (3.50) outside black holes follow directly from (2.13) and (2.14). We omit the detailed arguments here.

To prove Theorem 3.3, we then show that the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ have their restrictions on the sets $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$ being well-defined, invariant, invertible and measure-preserving.

Proposition 3.8. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$ and $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, the sets $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$ be as in (2.2) and (2.9), and let transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ be as in (2.13) and (2.14). Then the following statements hold.*

- (i) *The sets $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$ are invariant under the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$; i.e.,*

$$(3.51)$$

$$R_{a,b,c}\mathcal{D}_{a,b,c} = \tilde{R}_{a,b,c}\mathcal{D}_{a,b,c} = \mathcal{D}_{a,b,c} \text{ and } R_{a,b,c}\mathcal{S}_{a,b,c} = \tilde{R}_{a,b,c}\mathcal{S}_{a,b,c} = \mathcal{S}_{a,b,c}.$$

- (ii) *The restriction of $\tilde{R}_{a,b,c}$ onto $\mathcal{S}_{a,b,c}$ (resp. $\mathcal{D}_{a,b,c}$) is the inverse of the restriction of $R_{a,b,c}$ onto $\mathcal{S}_{a,b,c}$ (resp. $\mathcal{D}_{a,b,c}$), and vice versa; i.e.,*

$$(3.52)$$

$$R_{a,b,c}(\tilde{R}_{a,b,c}(t)) = \tilde{R}_{a,b,c}(R_{a,b,c}(t)) = t \text{ for all } t \in \mathcal{S}_{a,b,c} \text{ (resp. } \mathcal{D}_{a,b,c}).$$

- (iii) *The restriction of the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ onto $\mathcal{S}_{a,b,c}$ (resp. $\mathcal{D}_{a,b,c}$) are measure-preserving transformations; i.e.,*

$$(3.53) \quad |R_{a,b,c}(E)| = |\tilde{R}_{a,b,c}(E)| = |E|$$

for any measurable set $E \subset \mathcal{S}_{a,b,c}$ (resp. $\mathcal{D}_{a,b,c}$).

We will use Proposition 3.8 to establish the uniqueness of solution $\mathbf{x} \in \mathcal{B}_b^0$ for the infinite-dimensional linear system $\mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{1}$, which is a pivotal observation and our starting point to explore various properties of the sets $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$ deeply.

Proposition 3.9. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$ and $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, and let the set $\mathcal{S}_{a,b,c}$ be as in (2.9). Then for any $t \in \mathcal{S}_{a,b,c}$ there exists a unique vector $\mathbf{x} \in \mathcal{B}_b^0$ such that $\mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{1}$.*

We will also apply Proposition 3.8 to show that the black hole of the piecewise linear transformation $\tilde{R}_{a,b,c}$ has empty intersection with the invariant set $\mathcal{S}_{a,b,c}$, which plays important roles in the explicit construction of the maximal invariant set $\mathcal{S}_{a,b,c}$ in Theorems 5.4 and 5.5.

Proposition 3.10. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$ and $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, the set $\mathcal{S}_{a,b,c}$ be as in (2.9), and let the transformation $R_{a,b,c}$ be as in (2.13). Then*

$$(3.54) \quad (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset \quad \text{for all } n \geq 0.$$

To prove Theorem 3.3, we finally need the following density property of the sets $\mathcal{S}_{a,b,c}$ and $\mathcal{D}_{a,b,c}$ around the origin, c.f. Lemma 5.12 for similar density property of the set $\mathcal{S}_{a,b,c}$ when the ratio between a and b is rational.

Lemma 3.11. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, $b - a < c_0 := c - \lfloor c/b \rfloor b < a$ and $a/b \notin \mathbb{Q}$, the piecewise linear transformation $R_{a,b,c}$ be as in (2.13), and let $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$ be as in (2.2) and (2.9) respectively. Then the following conclusions hold.*

- (i) *If $\mathcal{S}_{a,b,c} \neq \emptyset$, then*

$$(3.55) \quad (0, \epsilon) \cap \mathcal{S}_{a,b,c} \neq \emptyset \text{ and } (-\epsilon, 0) \cap \mathcal{S}_{a,b,c} \neq \emptyset$$

for any $\epsilon > 0$.

- (ii) *If $\mathcal{D}_{a,b,c} \neq \emptyset$, then*

$$(3.56) \quad (0, \epsilon) \cap \mathcal{D}_{a,b,c} \neq \emptyset \text{ and } (-\epsilon, 0) \cap \mathcal{D}_{a,b,c} \neq \emptyset$$

for any $\epsilon > 0$.

We prove Theorem 3.3 by assuming that Proposition 3.8 and Lemma 3.11 hold.

Proof of Theorem 3.3. The necessity is obvious.

Now the sufficiency. By (3.2), (3.35), (3.36), (3.41), (3.42), and Theorems 2.1 and 2.2, it remains to prove the sufficiency for the cases that $b - a < c_0 < a$ and $a/b \notin \mathbb{Q}$ and that $b - a < c_0 < a$ and $a/b \in \mathbb{Q}$.

Case 1: $b - a < c_0 < a$ and $a/b \notin \mathbb{Q}$.

Suppose on the contrary that $\mathcal{D}_{a,b,c} \neq \emptyset$. Then by (3.2), Lemma 3.11, and the assumption that $\mathcal{D}_{a,b,c} \cap \mathcal{Z}_{a,b} = \emptyset$, there exist $t_n \in \mathcal{D}_{a,b,c} \cap (0, a)$, $n \geq 1$ such that $\{t_n\}_{n=1}^\infty$ is a decreasing sequence that converges to zero. Following the argument used in the proof of Lemma 3.4, we obtain that $0 \in \mathcal{D}_{a,b,c}$. This leads to the contradiction.

Case 2: $b - a < c_0 < a$ and $a/b \in \mathbb{Q}$.

Write $a/b = p/q$ for some coprime integers p and q . Suppose on the contrary that $\mathcal{D}_{a,b,c} \neq \emptyset$; i.e., $\mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{2}$ for some $t \in \mathbb{R}$ and $\mathbf{x} \in \mathcal{B}_b^0$. By (3.2) and the assumption that $\mathcal{D}_{a,b,c} \cap \mathcal{Z}_{a,b} = \emptyset$, we may assume that $t \in [0, pb/q) \setminus \{0, b/q, \dots, b(p-1)/q\}$ without loss of generality. If $0 \neq t - \lfloor qt/b \rfloor b/q < c - \lfloor qc/b \rfloor b/q$, then $\chi_{[0,c)}(\lfloor qt/b \rfloor b/q - \mu + \lambda) = \chi_{[0,c)}(t - \mu + \lambda)$ for all $\mu \in a\mathbb{Z}$ and $\lambda \in b\mathbb{Z}$. This implies that

$$\mathbf{M}_{a,b,c}(\lfloor qt/b \rfloor b/q)\mathbf{x} = \mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{2},$$

and hence $\lfloor qt/b \rfloor b/q \in \mathcal{D}_{a,b,c} \cap \mathcal{Z}_{a,b}$, which is a contradiction.

If $0 \neq t - \lfloor qt/b \rfloor b/q \geq c - \lfloor qc/b \rfloor b/q$, then $\chi_{(0,c]}(\lfloor qt/b \rfloor b/q + 1)b/q - \mu + \lambda) = \chi_{[0,c)}(t - \mu + \lambda)$ for all $\mu \in a\mathbb{Z}$ and $\lambda \in b\mathbb{Z}$. Hence $\mathbf{M}_{a,b,c}(c - (\lfloor qt/b \rfloor + 1)b/q)\tilde{\mathbf{x}} = \mathbf{2}$, where $\tilde{\mathbf{x}} = (\mathbf{x}(-\lambda))_{\lambda \in b\mathbb{Z}} \in \mathcal{B}_b^0$. This contradicts to the assumption that $\mathcal{D}_{a,b,c} \cap (c - \mathcal{Z}_{a,b}) = \emptyset$. \square

We finish this subsection with proofs of Proposition 3.8, Lemma 3.11, and Propositions 3.9 and 3.10.

Proof of Proposition 3.8. (i) By (3.1) and Proposition 3.6, we have that

$$(3.57) \quad R_{a,b,c}\mathcal{S}_{a,b,c} \subset \mathcal{S}_{a,b,c}$$

and

$$(3.58) \quad \tilde{R}_{a,b,c}\mathcal{S}_{a,b,c} \subset \mathcal{S}_{a,b,c}.$$

Observe that

$$(3.59) \quad \tilde{R}_{a,b,c}(R_{a,b,c}(t)) = R_{a,b,c}(\tilde{R}_{a,b,c}(t)) = t \quad \text{for all } t \in \mathcal{S}_{a,b,c}$$

by Propositions 3.6 and 3.7. Hence

$$(3.60) \quad R_{a,b,c}\mathcal{S}_{a,b,c} = \tilde{R}_{a,b,c}\mathcal{S}_{a,b,c} = \mathcal{S}_{a,b,c}$$

by (3.57), (3.58) and (3.59).

Let $t \in \mathcal{D}_{a,b,c}$ and $\mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{2}$ for some $\mathbf{x} \in \mathcal{B}_b^0$. By (3.13), there is a decomposition $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1$ such that $\mathbf{x}_0 \in \mathcal{B}_b^0$, $\mathbf{x}_1 \in \mathcal{B}_b$ and $\mathbf{M}_{a,b,c}(t)\mathbf{x}_0 = \mathbf{M}_{a,b,c}(t)\mathbf{x}_1 = \mathbf{1}$. Therefore $\tau_{\lambda_{a,b,c}(t)}\mathbf{x} \in \mathcal{B}_b^0$ by Proposition 3.6 and $\mathbf{M}_{a,b,c}(R_{a,b,c}(t))\tau_{\lambda_{a,b,c}(t)}\mathbf{x} = \mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{2}$. This proves that

$$(3.61) \quad R_{a,b,c}\mathcal{D}_{a,b,c} \subset \mathcal{D}_{a,b,c}.$$

Similarly we have that

$$(3.62) \quad \tilde{R}_{a,b,c}\mathcal{D}_{a,b,c} \subset \mathcal{D}_{a,b,c}.$$

Recalling that $\mathcal{D}_{a,b,c} \subset \mathcal{S}_{a,b,c}$ and combining (3.59), (3.61) and (3.62), we obtain

$$(3.63) \quad R_{a,b,c}\mathcal{D}_{a,b,c} = \tilde{R}_{a,b,c}\mathcal{D}_{a,b,c} = \mathcal{D}_{a,b,c}.$$

Therefore (3.51) follows from (3.60) and (3.63).

(ii) The invertibility of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ in the second conclusion follows from (3.30) and (3.50).

(iii) By (2.13),

$$(3.64) \quad |R_{a,b,c}(E)| \leq |E|$$

for any measurable set E , and the above inequality becomes an equality,

$$(3.65) \quad |R_{a,b,c}(E)| = |E|,$$

whenever E has empty intersection with the black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$ of the transformation $R_{a,b,c}$. This, together with (3.27), (3.30) and the first conclusion, proves that $R_{a,b,c}$ is a measure-preserving transformation on the sets $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$.

The measure-preserving property for the transformation $\tilde{R}_{a,b,c}$ can be established by applying similar argument. \square

Proof of Lemma 3.11. (i): Let $t_n := (R_{a,b,c})^n(t_0)$, $n \geq 0$, be the orbit of the transformation $R_{a,b,c}$ with initial $t_0 \in \mathcal{S}_{a,b,c}$, and set $\tilde{t}_n := t_n - \lfloor t_n/a \rfloor a$, $n \geq 0$. Without loss of generality, we assume that $\tilde{t}_n \neq 0$ for all $n \geq 0$, because at most one in the sequence \tilde{t}_n , $n \geq 0$, could be zero by the assumption $a/b \notin \mathbb{Q}$ and in that case we replace the initial t_0 by t_{n_0} for some sufficiently large n_0 . Clearly the proof of the first non-empty-set conclusion in (3.55) reduces to proving that for sufficiently small $\epsilon > 0$ there exists an index n with the property that

$$(3.66) \quad \tilde{t}_n \in (0, \epsilon).$$

From (3.3), Propositions 3.6 and 3.8, it follows that $\tilde{t}_n \in \mathcal{S}_{a,b,c} \cap ([0, c_0 + a - b) \cap [c_0, a))$, $n \in \mathbb{Z}_+$. Recall that for $n \neq m$, $t_n - t_m = kb$ for some $0 \neq k \in \mathbb{Z}$ by (2.13), which together with $a/b \notin \mathbb{Q}$ implies that

$$(3.67) \quad \tilde{t}_n - \tilde{t}_m \neq 0 \text{ whenever } n \neq m.$$

As $\tilde{t}_n, n \geq 0$, lie in the bounded set $(0, a)$, there exist integers $n_1 < n_2$ such that $|\tilde{t}_{n_1} - \tilde{t}_{n_2}| < \epsilon$. Without loss of generality, we assume that $\tilde{t}_{n_1} \notin (0, \epsilon)$, otherwise the conclusion (3.66) holds by letting $n = n_1$. Recall that $\tilde{t}_n \in [0, c_0 + a - b) \cap [c_0, a)$ for all $n \geq 0$, we have that either $\tilde{t}_{n_1}, \tilde{t}_{n_2} \in [0, c_0 + a - b)$ or $\tilde{t}_{n_1}, \tilde{t}_{n_2} \in [c_0, a)$. This implies that $\lambda_{a,b,c}(t_{n_1}) = \lambda_{a,b,c}(t_{n_2})$, where $\lambda_{a,b,c}(t)$ is defined in (3.28). If $\lambda_{a,b,c}(t_{n_1+k}) = \lambda_{a,b,c}(t_{n_2+k})$ for all positive integers k , then $t_{n_1+k} - t_{n_2+k} = t_{n_1} - t_{n_2}$ for all positive integers k . Applying the above equality with $k = (n_2 - n_1)l, l \in \mathbb{N}$, we obtain that $t_{n_1+l(n_2-n_1)} = t_{n_1} + l(t_{n_2} - t_{n_1})$ for all $l \in \mathbb{N}$. Therefore $\tilde{t}_{n_1+l_0(n_2-n_1)} = \tilde{t}_{n_1} + l_0(\tilde{t}_{n_2} - \tilde{t}_{n_1}) \in (0, \epsilon)$ for $l_0 = \lfloor \tilde{t}_{n_1}/(\tilde{t}_{n_1} - \tilde{t}_{n_2}) \rfloor$ if $\tilde{t}_{n_2} - \tilde{t}_{n_1} < 0$, and $\tilde{t}_{n_1+(l_1+1)(n_2-n_1)} = \tilde{t}_{n_1} + (l_1+1)(\tilde{t}_{n_2} - \tilde{t}_{n_1}) - a \in (0, \epsilon)$ for $l_1 = \lfloor (a - \tilde{t}_{n_1})/(\tilde{t}_{n_2} - \tilde{t}_{n_1}) \rfloor$ if $\tilde{t}_{n_2} - \tilde{t}_{n_1} > 0$. This together with (3.67) proves that (3.66) holds when $\lambda_{a,b,c}(t_{n_1+k}) = \lambda_{a,b,c}(t_{n_2+k})$ for all positive integers k . Now we consider the case that there exists a positive integer m such that $\lambda_{a,b,c}(t_{n_1+k}) = \lambda_{a,b,c}(t_{n_2+k})$ for all nonnegative integer $k \in [0, m-1]$ and $\lambda_{a,b,c}(t_{n_1+m}) \neq \lambda_{a,b,c}(t_{n_2+m})$. Then we can prove by induction that $t_{n_1+k} - t_{n_2+k} = t_{n_1} - t_{n_2}$ for all $0 \leq k \leq m-1$. This implies that either $|\tilde{t}_{n_1+m-1} - \tilde{t}_{n_2+m-1}| = |\tilde{t}_{n_1} - \tilde{t}_{n_2}|$ or $|\tilde{t}_{n_1+m-1} - \tilde{t}_{n_2+m-1}| + |\tilde{t}_{n_1} - \tilde{t}_{n_2}| = a$. On the other hand, it follows from $\lambda_{a,b,c}(t_{n_1+m}) \neq \lambda_{a,b,c}(t_{n_2+m})$ that $\tilde{t}_{n_1+m-1}, \tilde{t}_{n_2+m-1}$ should lie in the different intervals $[0, c_0 + a - b)$ and $[c_0, a)$. Therefore $|\tilde{t}_{n_1+m-1} - \tilde{t}_{n_2+m-1}| + |\tilde{t}_{n_1} - \tilde{t}_{n_2}| = a$, which implies that one and only one of $\tilde{t}_{n_1+(m-1)}, \tilde{t}_{n_2+(m-1)}$ belongs to $(0, \epsilon)$, while the other one of $\tilde{t}_{n_1+(m-1)}, \tilde{t}_{n_2+(m-1)}$ belongs to $[a - \epsilon, a)$. Hence (3.66) holds when $\lambda_{a,b,c}(t_{n_1+m}) = \lambda_{a,b,c}(t_{n_2+m})$ for some positive integers m and we complete the proof of the conclusion (3.66).

The second conclusion in (3.55) can be proved by using similar argument with l_0 replaced by $l_0 + 1, l_1 + 1$ by l_1 .

(ii): We can follow the argument of the first conclusion line by line except with the set $\mathcal{S}_{a,b,c}$ replaced by its subset $\mathcal{D}_{a,b,c}$. We omit the detailed arguments here. \square

Proof of Proposition 3.9. Suppose on the contrary that $\mathbf{M}_{a,b,c}(t)\mathbf{x}_0 = \mathbf{M}_{a,b,c}(t)\mathbf{x}_1 = \mathbf{1}$ for two distinct vectors $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{B}_b^0$. Then there exists $0 \neq \lambda_0 \in b\mathbb{Z}$ such that $\mathbf{x}_0(\lambda_0) \neq \mathbf{x}_1(\lambda_0)$ while $\mathbf{x}_0(\lambda) = \mathbf{x}_1(\lambda)$ for all $|\lambda| < |\lambda_0|$. Without loss of generality, we assume that $\mathbf{x}_0(\lambda_0) = 1$ and $\mathbf{x}_1(\lambda_0) = 0$. Let $\lambda_1 \in b\mathbb{Z}$ be so chosen that $\lambda_1\lambda_0 \geq 0, |\lambda_1| < |\lambda_0|$ and $|\lambda_1 - \lambda_0| = \min_{\lambda \in (-|\lambda_0|, |\lambda_0|) \cap b\mathbb{Z}, \mathbf{x}_0(\lambda) = \mathbf{x}_1(\lambda) = 1} |\lambda - \lambda_0|$. Thus $\mathbf{M}_{a,b,c}(t - \lambda_1)\tau_{\lambda_1}\mathbf{x}_0 = \mathbf{M}_{a,b,c}(t - \lambda_1)\tau_{\lambda_1}\mathbf{x}_1 = \mathbf{1}$ by (3.2), and both $\tau_{\lambda_1}\mathbf{x}_0$ and $\tau_{\lambda_1}\mathbf{x}_1$ belong to \mathcal{B}_b^0 by the selection of the index λ_1 . As $\lambda_0 - \lambda_1$ is the closest positive (or negative) index to zero such that $\tau_{\lambda_1}\mathbf{x}_0(\lambda_0 - \lambda_1) = 1$.

Then it follows from Proposition 3.6 that $\tau_{\lambda_1} \mathbf{x}_1(\lambda_0 - \lambda_1) = 1$, which contradicts to $\mathbf{x}_1(\lambda_0) = 0$. \square

Proof of Proposition 3.10. Suppose, on the contrary, that (3.54) does not hold. Then there exists a nonnegative integer m such that $(R_{a,b,c})^m([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} \neq \emptyset$ and $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset$ for all $0 \leq n < m$. We observe that $m \neq 0$ (or equivalently m is a positive integer) by Proposition 3.6. Take $t \in (R_{a,b,c})^m([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c}$. Then $t = R_{a,b,c}(s)$ for some $s \in (R_{a,b,c})^{m-1}([c - c_0, c - c_0 + b - a) + a\mathbb{Z})$. If $s \in [c_0 + a - b, c_0) + a\mathbb{Z}$, then $t = s$ by (2.13), which is a contradiction as $([c_0 + a - b, c_0) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset$ by Proposition 3.6. If $s \notin [c_0 + a - b, c_0) + a\mathbb{Z}$, then $s = \tilde{R}_{a,b,c}(t) \in \mathcal{S}_{a,b,c}$ by (3.50) and Proposition 3.8. Hence $s \in (R_{a,b,c})^{m-1}([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c}$, which is a contradiction. \square

4. MAXIMAL INVARIANT SETS OF PIECEWISE LINEAR TRANSFORMATIONS

In this section, we prove Theorem 2.3. To do so, we need the most crucial observation of this paper that $\mathcal{S}_{a,b,c}$ is the maximal set such that

$$(4.1) \quad R_{a,b,c}\mathcal{S}_{a,b,c} \subset \mathcal{S}_{a,b,c} \quad \text{and} \quad \tilde{R}_{a,b,c}\mathcal{S}_{a,b,c} \subset \mathcal{S}_{a,b,c},$$

and

$$(4.2) \quad \mathcal{S}_{a,b,c} \cap ([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) = \mathcal{S}_{a,b,c} \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) = \emptyset,$$

c.f. Lemma 5.11 for the case that $a/b \in \mathbb{Q}$.

Theorem 4.1. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$ and $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, the set $\mathcal{S}_{a,b,c}$ be as in (2.9), and let transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ be as in (2.13) and (2.14). Then*

- (i) $\mathcal{S}_{a,b,c}$ is the maximal set that is invariant under the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$, and that has empty intersection with their black holes.
- (ii) The complement of $\mathcal{S}_{a,b,c}$ is the minimal set that is invariant under the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$, and that contains their black holes.

To prove Theorem 2.3 (and the characterization of (2.11) in Theorem 5.7), we need the following deep connection between the invariant sets $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$.

Theorem 4.2. *Let (a, b, c) be a triple of positive numbers with $a < b < c$ and $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, and let the sets $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$ be as in (2.2) and (2.9). Then*

$$(4.3) \quad \begin{aligned} \mathcal{D}_{a,b,c} = & (\mathcal{S}_{a,b,c} \cap ([0, c_0 + a - b] + a\mathbb{Z}) \cap (\mathcal{S}_{a,b,c} - \lfloor c/b \rfloor b)) \\ & \cup (\mathcal{S}_{a,b,c} \cap (\cup_{\lambda \in [b, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}} (\mathcal{S}_{a,b,c} - \lambda))). \end{aligned}$$

In next two subsections, we prove Theorems 4.1 and 4.2, and apply them to prove Theorem 2.3.

4.1. Maximal invariant sets. To establish maximality in Theorem 4.1, we characterize whether a particular point belongs to the set $\mathcal{S}_{a,b,c}$, which will also be used in the proofs of Theorems 5.4 and 5.5.

Proposition 4.3. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$ and $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, and let $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ be as in (2.13) and (2.14). Then $t \in \mathcal{S}_{a,b,c}$ if and only if $(R_{a,b,c})^n(t)$ and $(\tilde{R}_{a,b,c})^n(t), n \geq 0$, do not belong to the black holes of the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$.*

Now assuming that Proposition 4.3 holds, we start to prove Theorem 4.1.

Proof of Theorem 4.1. (i) By (3.30) in Proposition 3.6, and (3.51) in Proposition 3.8, the set $\mathcal{S}_{a,b,c}$ is an invariant set under piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ which has empty intersection with their black holes. Then it suffices to show the maximality. Let E be an invariant set under piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ that has empty intersection with their black holes. Take $t \in E$. Then it follows from the invariance of the set E that

$$(4.4) \quad (R_{a,b,c})^n(t) \in E \quad \text{and} \quad (\tilde{R}_{a,b,c})^n(t) \in E$$

for all nonnegative integers n . This, together with Proposition 4.3 and the empty intersection assumption between the set E and the black holes of the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$, implies that $t \in \mathcal{S}_{a,b,c}$. Thus $E \subset \mathcal{S}_{a,b,c}$ and the maximality of the set $\mathcal{S}_{a,b,c}$ follows.

(ii) By (3.30) in Proposition 3.6 and (3.51) in Proposition 3.8, the complement of $\mathcal{S}_{a,b,c}$ is an invariant set under the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$, and it contains their black holes. For any set E that is invariant under the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ that contains their black holes, we obtain from (3.50) that $R_{a,b,c}(t), \tilde{R}_{a,b,c}(t) \notin E$ for any $t \notin E$. This, together with Proposition 4.3, proves the desired minimality for the complement of $\mathcal{S}_{a,b,c}$. \square

Next we prove Proposition 4.3.

Proof of Proposition 4.3. (\implies) Take $t \in \mathcal{S}_{a,b,c}$. Then $(R_{a,b,c})^n(t) \in \mathcal{S}_{a,b,c}$ and $(\tilde{R}_{a,b,c})^n(t) \in \mathcal{S}_{a,b,c}$ for all $n \geq 0$ by Proposition 3.8. This together with Proposition 3.6 proves the desired empty intersection property with black holes.

(\impliedby) Take any real number t such that $(R_{a,b,c})^n(t)$ and $(\tilde{R}_{a,b,c})^n(t)$, $n \geq 0$, do not belong to the black holes of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$. Define

$$t_n = \begin{cases} (R_{a,b,c})^n(t) & \text{if } n \geq 1 \\ t & \text{if } n = 0 \\ (\tilde{R}_{a,b,c})^{-n}(t) & \text{if } n < 0, \end{cases}$$

and $\lambda_n = t_n - t$, $n \in \mathbb{Z}$. Then

$$(4.5) \quad t_{n+m} = (R_{a,b,c})^m(t_n) \quad \text{for all } n \in \mathbb{Z} \text{ and } 0 \leq m \in \mathbb{Z}$$

and

$$(4.6) \quad \lambda_n \in b\mathbb{Z} \text{ and } \lambda_{n+1} - \lambda_n \in \{[c/b]b + b, [c/b]b\} \quad \text{for all } n \in \mathbb{Z},$$

by (3.50), and the assumption that $t_n \notin [c_0 + a - b, c_0) + a\mathbb{Z}$ for all $n \geq 0$ and $t_n \notin [c - c_0, c - c_0 + b - a) + a\mathbb{Z}$ for all $n \leq 0$. Define $\mathbf{x}_t(\lambda) = 1$ if $\lambda = \lambda_n$ for some $n \in \mathbb{Z}$ and $\mathbf{x}_t(\lambda) = 0$ otherwise, and set $\mathbf{x}_t = (\mathbf{x}_t(\lambda))_{\lambda \in b\mathbb{Z}}$. Then \mathbf{x}_t belong to \mathcal{B}_b^0 . Let $\mu_n \in a\mathbb{Z}$ be so chosen that $\tilde{t}_n := t_n - \mu_n \in [0, a)$. Then $\{\mu_n\}_{n \in \mathbb{Z}}$ is a strictly increasing sequence with

$$(4.7) \quad \lim_{n \rightarrow +\infty} \mu_n = +\infty \text{ and } \lim_{n \rightarrow -\infty} \mu_n = -\infty$$

by (4.6), and

$$(4.8) \quad \begin{aligned} & \sum_{\lambda \in b\mathbb{Z}} \chi_{[0,c)}(t - \mu_n + \lambda) \mathbf{x}_t(\lambda) = \sum_{m \in \mathbb{Z}} \chi_{[0,c)}(t - \mu_n + \lambda_m) \\ & = \sum_{m \in \mathbb{Z}} \chi_{[0,c)}(t_m - \mu_n) = \chi_{[0,c)}(t_n - \mu_n) = 1 \quad \text{for all } n \in \mathbb{Z}, \end{aligned}$$

where the first equation follows from the definition of the vector \mathbf{x}_t and the third one holds as $t_m - \mu_n \leq t_n - \mu_n - b < 0$ for all $m < n$ and $t_m - \mu_n \geq (t_{n+1} - t_n) + (t_n - \mu_n) = (\lambda_{n+1} - \lambda_n) + (t_n - \mu_n) \geq c$ for all $m > n$. Similarly for any $\mu \in a\mathbb{Z}$ with $\mu_n < \mu < \mu_{n+1}$,

$$(4.9) \quad \sum_{\lambda \in b\mathbb{Z}} \chi_{[0,c)}(t - \mu + \lambda) \mathbf{x}_t(\lambda) = \sum_{m \in \mathbb{Z}} \chi_{[0,c)}(t_m - \mu) = 1$$

as $t_m - \mu \leq t_n - \mu < \mu_n + a - \mu \leq 0$ for $m \leq n$, $0 \leq t_{n+1} - \mu_{n+1} < t_m - \mu \leq t_{n+1} - \mu_n - a < c$ for $m = n+1$, and $t_m - \mu \geq t_{n+2} - \mu_{n+1} + a \geq c$

for $m \geq n + 2$. Combining (4.8) and (4.9) proves $\mathbf{M}_{a,b,c}(t)\mathbf{x}_t = \mathbf{1}$ and hence $t \in \mathcal{S}_{a,b,c}$. \square

We conclude this subsection by the proof of Theorem 4.2.

Proof of Theorem 4.2. We use the double inclusion method. Take $t \in \mathcal{D}_{a,b,c}$. Let $\mathbf{x} \in \mathcal{B}_b^0$ be so chosen that $\mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{2}$ and denote by K the support of the vector \mathbf{x} ; i.e., the set of all $\lambda \in b\mathbb{Z}$ with $\mathbf{x}(\lambda) = 1$. Write $K = \{\lambda_j : j \in \mathbb{Z}\}$ for some strictly increasing sequence $\{\lambda_j\}_{j=-\infty}^{\infty}$ in $b\mathbb{Z}$ with $\lambda_0 = 0$, and define $\mathbf{x}_i := (\mathbf{x}_i(\lambda))_{\lambda \in a\mathbb{Z}}$, $i = 0, 1$, by $\mathbf{x}_i(\lambda) = 1$ if $\lambda \in K_i$ and $\mathbf{x}_i(\lambda) = 0$ otherwise, where $K_i = \{\lambda_{i+2j} : j \in \mathbb{Z}\}$, $i = 0, 1$. Then following the argument in the proof of the implication (ii) \implies (iii) of Theorem 3.2, we have that

$$(4.10) \quad \mathbf{x}_0, \tau_{\lambda_1}\mathbf{x}_1 \in \mathcal{B}_b^0 \quad \text{and} \quad \mathbf{M}_{a,b,c}(t)\mathbf{x}_0 = \mathbf{M}_{a,b,c}(t + \lambda_1)\tau_{\lambda_1}\mathbf{x}_1 = \mathbf{1}.$$

Notice that $\mathcal{D}_{a,b,c} \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) = \emptyset$ by the supset property $\mathcal{D}_{a,b,c} \subset \mathcal{S}_{a,b,c}$ in (3.27) and the trivial intersection property $\mathcal{S}_{a,b,c} \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) = \emptyset$ in (3.30). Then either $t \in [0, c_0 + a - b) + a\mathbb{Z}$ or $[c_0, a) + a\mathbb{Z}$. For the first case that $t \in [0, c_0 + a - b) + a\mathbb{Z}$, $\lambda_2 = \lfloor c/b \rfloor b + b$ by (3.28) and (3.31) in Proposition 3.6. Hence $\lambda_1 \in [b, \lfloor c/b \rfloor b] \cap b\mathbb{Z}$ and $t + \lambda_1 \in \mathcal{S}_{a,b,c}$ by (4.10). Thus

$$(4.11) \quad t \in (\mathcal{S}_{a,b,c} \cap ([0, c_0 + a - b) + a\mathbb{Z})) \cap (\cup_{\lambda_1 \in [b, \lfloor c/b \rfloor b] \cap b\mathbb{Z}} (\mathcal{S}_{a,b,c} - \lambda_1))$$

for the first case. Similarly for the second case that $t \in [c_0 + a - b, a) + a\mathbb{Z}$, $\lambda_2 = \lfloor c/b \rfloor b$ by (3.28) and (3.31) in Proposition 3.6, which together with (4.10) implies that

$$(4.12) \quad t \in (\mathcal{S}_{a,b,c} \cap ([c_0, a) + a\mathbb{Z})) \cap (\cup_{\lambda_1 \in [b, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}} (\mathcal{S}_{a,b,c} - \lambda_1))$$

for the second case. Combining (4.11) and (4.12) proves the first inclusion

$$(4.13) \quad \begin{aligned} \mathcal{D}_{a,b,c} &\subset (\mathcal{S}_{a,b,c} \cap ([0, c_0 + a - b) + a\mathbb{Z})) \cap (\mathcal{S}_{a,b,c} - \lfloor c/b \rfloor b) \\ &\cup (\mathcal{S}_{a,b,c} \cap (\cup_{\lambda \in [b, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}} (\mathcal{S}_{a,b,c} - \lambda))). \end{aligned}$$

Take $t \in \mathcal{S}_{a,b,c} \cap ([0, c_0 + a - b) + a\mathbb{Z}) \cap (\mathcal{S}_{a,b,c} - \lfloor c/b \rfloor b)$. Then there exist $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{B}_b^0$ such that

$$(4.14) \quad \mathbf{M}_{a,b,c}(t)\mathbf{x}_0 = \mathbf{M}_{a,b,c}(t + \lfloor c/b \rfloor b)\mathbf{x}_1 = \mathbf{1}.$$

Define $\mathbf{x} = \mathbf{x}_0 + \tau_{-\lfloor c/b \rfloor b}\mathbf{x}_1$. By (3.1) and (4.14), we have that

$$(4.15) \quad \mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{M}_{a,b,c}(t)\mathbf{x}_0 + \mathbf{M}_{a,b,c}(t + \lfloor c/b \rfloor b)\mathbf{x}_1 = \mathbf{2}.$$

Now let us verify that $\mathbf{x} \in \mathcal{B}_b^0$. Write $\mathbf{x} = (\mathbf{x}(\lambda))_{\lambda \in b\mathbb{Z}}$. Observe that $\mathbf{x}(\lambda) \in \{0, 1, 2\}$ for all $\lambda \in b\mathbb{Z}$ and $\mathbf{x}(0) \geq \mathbf{x}_0(0) \geq 1$. Then it suffices to prove that $\mathbf{x}(\lambda) \neq 2$ for all $\lambda \in b\mathbb{Z}$. Suppose, to the contrary, that

$\mathbf{x}(\lambda_0) = 2$ for some $\lambda_0 \in b\mathbb{Z}$. Then $\mathbf{x}_0(\lambda_0) = 1$ and $\tau_{-\lfloor c/b \rfloor b} \mathbf{x}_1(\lambda_0) = 1$. Hence $\tau_{\lambda_0} \mathbf{x}_0, \tau_{\lambda_0 - \lfloor c/b \rfloor b} \mathbf{x}_1 \in \mathcal{B}_b^0$ and

$\mathbf{M}_{a,b,c}(t + \lambda_0) \tau_{\lambda_0} \mathbf{x}_0 = \mathbf{M}_{a,b,c}(t) \mathbf{x}_0 = \mathbf{1}$ and $\mathbf{M}_{a,b,c}(t + \lambda_0) \tau_{\lambda_0 - \lfloor c/b \rfloor b} \mathbf{x}_1 = \mathbf{1}$ by (3.1) and (4.14). Thus $\tau_{\lambda_0} \mathbf{x}_0 = \tau_{\lambda_0 - \lfloor c/b \rfloor b} \mathbf{x}_1$ by Proposition 3.9, which is a contradiction as $\tau_{-\lfloor c/b \rfloor b} \mathbf{x}_1(\lfloor c/b \rfloor b) = \mathbf{x}_1(0) = 1$ by the assumption that $\mathbf{x}_1 \in \mathcal{B}_b^0$ and $\mathbf{x}_0(\lfloor c/b \rfloor b) = 0$ by (3.31) and the assumption that $t \in [0, c_0 + b - a) \cap \mathcal{S}_{a,b,c}$. Therefore \mathbf{x} is a binary vector in \mathcal{B}_b^0 . This together with (4.15) proves that

$$(4.16) \quad \mathcal{S}_{a,b,c} \cap ([0, c_0 + a - b) + a\mathbb{Z}) \cap (\mathcal{S}_{a,b,c} - \lfloor c/b \rfloor b) \subset \mathcal{D}_{a,b,c}.$$

Applying similar argument, we can prove that

$$(4.17) \quad \mathcal{S}_{a,b,c} \cap ([0, c_0 + a - b) + a\mathbb{Z}) \cap (\mathcal{S}_{a,b,c} - \lambda) \subset \mathcal{D}_{a,b,c}$$

for all $\lambda \in [b, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}$, and

$$(4.18) \quad \mathcal{S}_{a,b,c} \cap ([c_0, a) + a\mathbb{Z}) \cap (\mathcal{S}_{a,b,c} - \lambda) \subset \mathcal{D}_{a,b,c}$$

for all $\lambda \in [b, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}$. The desired equality (4.3) then follows from (4.13), (4.16), (4.17) and (4.18). \square

4.2. Proof of Theorem 2.3. The main ideas behind the proof of Theorem 2.3 are as follows. The conclusion (VIII) follows from the results in [25, Section 3.3.3.5, 3.3.3.6 and 3.3.4.3]. We include a different proof by showing that the only binary vector \mathbf{x} satisfying $\mathbf{M}_{a,b,c}(t)\mathbf{x} = \mathbf{2}$ is the vector $\mathbf{1}$. We prove Conclusion (IX) by showing that any point not in the black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$ of the transformation $R_{a,b,c}$ is contained in the hole $(R_{a,b,c})^n[c - c_0, c - c_0 + b - a) + a\mathbb{Z}$ for some $n \geq 1$, see (4.21). The crucial step in the proof of Conclusion (X) (resp. Conclusion (XI)) is to show that $\mathcal{S}_{a,b,c} = [0, c_0 + a - b) + a\mathbb{Z}$ in (4.24) (resp. $\mathcal{S}_{a,b,c} = [c_0, a) + a\mathbb{Z}$ in (4.25)).

Proof of Theorem 2.3. (VIII): Suppose on the contrary that $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame. Then by Theorem 3.2 there exist $t \in \mathbb{R}$ and $(\mathbf{x}(\lambda))_{\lambda \in b\mathbb{Z}} \in \mathcal{B}_b^0$ such that

$$(4.19) \quad \sum_{\lambda \in b\mathbb{Z}} \chi_{[0,c)}(t - \mu + \lambda) \mathbf{x}(\lambda) = 2 \text{ for all } \mu \in a\mathbb{Z}.$$

By the assumption $\lfloor c/b \rfloor = 1$ and $b < c$, given any $t \in \mathbb{R}$ and $\mu \in a\mathbb{Z}$, the equality $\chi_{[0,c)}(t - \mu + \lambda) = 1$ holds for at most two distinct $\lambda \in b\mathbb{Z}$. This together with (4.19) that $\mathbf{x}(\lambda) = 1$ for all $\lambda \in b\mathbb{Z}$, and also that

$$(4.20) \quad t - \mu \notin [c, 2b) + b\mathbb{Z} \text{ for all } \mu \in a\mathbb{Z}.$$

If $a/b \notin \mathbb{Q}$, then there exists $\mu_0 \in a\mathbb{Z}$ by the density of the set $a\mathbb{Z} + b\mathbb{Z}$ in \mathbb{R} such that $t - \mu_0 \in [c, 2b) + b\mathbb{Z}$, which contradicts to (4.20).

If $a/b \in \mathbb{Q}$, then $a/b = p/q$ for some positive coprime integers p and q . Hence

$$t \notin [c, 2b) + b\mathbb{Z}/q = \mathbb{R},$$

where the first conclusion follows from (4.20) and the equality holds as $2b - c = b - c_0 > b - a \geq b/q$ by the assumption $0 < b - a < c_0 < a$. This is a contradiction.

(IX): By the supset property (3.27) and Theorem 3.2, it suffices to prove the following result.

Proposition 4.4. *Let (a, b, c) be a triple of positive numbers that satisfies $a < b < c, b - a < c_0 := c - \lfloor c/b \rfloor b < a, \lfloor c/b \rfloor \geq 2, c_1 = c - c_0 - \lfloor (c - c_0)/a \rfloor a > 2a - b$, and let $\mathcal{S}_{a,b,c}$ be as in (2.9). Then $\mathcal{S}_{a,b,c} = \emptyset$.*

Proof. By Propositions 3.6 and 3.10, it suffices to prove

$$(4.21) \quad [c_0 - a, c_0 + a - b) + a\mathbb{Z} \subset \cup_{n=0}^L (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}),$$

where $L = \max(\lfloor (c_0 + a - b)/(c_1 + b - 2a) \rfloor, \lfloor (a - c_0)/(a - c_1) \rfloor)$.

For any $t \in [0, c_0 + a - b)$, write $t = l(c_1 + b - 2a) + t'$ for some $t' \in [0, \min(c_1 + b - 2a, c_0 + a - b))$ and $0 \leq l \leq L$. Then

$$(4.22) \quad \begin{aligned} t &\in (R_{a,b,c})^l(t') + a\mathbb{Z} \subset (R_{a,b,c})^l([0, c_1 + b - 2a) + a\mathbb{Z}) \\ &\subset \cup_{n=0}^L (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \end{aligned}$$

for all $t \in [0, c_0 + a - b)$.

Similarly for any $s \in [c_0 - a, 0)$, let $s = l'(c_1 - a) + s'$ for some $s' \in [\max(c_1 - a, c_0 - a), 0)$ and $0 \leq l' \leq L$. Then

$$(4.23) \quad \begin{aligned} s &\in (R_{a,b,c})^{l'}(s') + a\mathbb{Z} \subset (R_{a,b,c})^{l'}([c_1 - a, 0) + a\mathbb{Z}) \\ &\subset \cup_{n=0}^L (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \end{aligned}$$

for all $s \in [c_1 - a, 0)$. Combining (4.22) and (4.23) and applying the periodic property (3.3) proves (4.21). \square

(X): Mimicking the argument used to prove (4.23), we can show that

$$\cup_{n=0}^{\infty} (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) = [c_0 + a - b, a) + a\mathbb{Z},$$

which implies that the set $\mathcal{S}_{a,b,c}$ in (2.9) is given by

$$(4.24) \quad \mathcal{S}_{a,b,c} = [0, c_0 + a - b) + a\mathbb{Z}.$$

From the assumption on c_1 , the ratio a/b is rational. We write $a/b = p/q$ for some coprime integers p and q . Clearly $p \geq 2$ as $b - a < c_0 < a$.

By the assumption that $c_1 = 2a - b$, we have that $\lfloor c/b \rfloor + 1 \in p\mathbb{Z}$, which implies that

$$R_{a,b,c}(t) - t \in a\mathbb{Z} \quad \text{for all } t \in \mathcal{S}_{a,b,c} = [0, c_0 + a - b) + a\mathbb{Z}.$$

This together with Theorem 3.2 and Proposition 4.2 implies that the Gabor system $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a frame of L^2 if and only if $\mathcal{D}_{a,b,c}$ in (2.2) is an empty set if and only if $([0, c_0 + a - b) + a\mathbb{Z}) \cap ([0, c_0 + a - b) + \lambda + a\mathbb{Z}) = \emptyset$ for all $\lambda \in [b, \lfloor c/b \rfloor b] \cap b\mathbb{Z}$. Observe that $([0, c_0 + a - b) + a\mathbb{Z}) \cap ([0, c_0 + a - b) + \lambda + a\mathbb{Z}) = [0, c_0 + a - b) + a\mathbb{Z} \neq \emptyset$ for $\lambda = pb \in [b, \lfloor c/b \rfloor b] \cap b\mathbb{Z}$ provided that $\lfloor c/b \rfloor \geq p$, and also that $([0, c_0 + a - b) + a\mathbb{Z}) \cap ([0, c_0 + a - b) + \lambda + a\mathbb{Z}) = [b/q, c_0 + a - b) + a\mathbb{Z} \neq \emptyset$ for $\lambda = kb \in [b, \lfloor c/b \rfloor b] \cap b\mathbb{Z}$ where $1 \leq k \leq p-1$ is the unique integer such that $qk-1 \in p\mathbb{Z}$, provided that $\lfloor c/b \rfloor + 1 = p$ and $c_0 + a - b > b/q$. Therefore the assumptions that $\lfloor c/b \rfloor + 1 = p$ and $c_0 + a - b \leq b/q$ are necessary for the Gabor system $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ being a frame of L^2 . On the other hand, if $\lfloor c/b \rfloor + 1 = p$ and $c_0 + a - b \leq b/q$, one may verify that $([0, c_0 + a - b) + a\mathbb{Z}) \cap ([0, c_0 + a - b) + \lambda + a\mathbb{Z}) = ([0, c_0 + a - b) + a\mathbb{Z}) \cap ([0, c_0 + a - b) + k(\lambda)b/q + a\mathbb{Z}) = \emptyset$ for all $\lambda \in [1, \lfloor c/b \rfloor]b \cap b\mathbb{Z}$, where $k(\lambda)$ is the unique integer in $[1, p-1]$ such that $k(\lambda)b/q - \lambda \in a\mathbb{Z}$. Therefore the assumptions that $\lfloor c/b \rfloor + 1 = p$ and $c_0 + a - b \leq b/q$ is also sufficient for the Gabor system $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ being a frame of L^2 .

(XI) Mimicking the argument used to prove (4.22), we may show that

$$(4.25) \quad \mathcal{S}_{a,b,c} = [c_0, a) + a\mathbb{Z}.$$

Now we can apply similar argument used in the proof of the conclusion (X) of this theorem. From the assumption that $c_1 = 0$, it follows $a/b = p/q$ for some coprime integers p and q with $p \geq 2$ and $\lfloor c/b \rfloor \in p\mathbb{Z}$. By (4.25) and Theorem 3.2, we can show that $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a frame of L^2 if and only if $([c_0, a) + a\mathbb{Z}) \cap ([c_0, a) + \lambda + a\mathbb{Z}) = \emptyset$ for all $\lambda \in [b, \lfloor c/b \rfloor b - b] \cap b\mathbb{Z}$. Then the desired necessary condition for the Gabor system $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ being a frame of L^2 follows from the observation that $([c_0, a) + a\mathbb{Z}) \cap ([c_0, a) + pb + a\mathbb{Z}) = [c_0, a) + a\mathbb{Z} \neq \emptyset$ if $\lfloor c/b \rfloor \geq p+1$, and that $([c_0, a) + a\mathbb{Z}) \cap ([c_0, a) + kb + a\mathbb{Z}) = [c_0, a - b/q) + a\mathbb{Z} \neq \emptyset$ if $\lfloor c/b \rfloor = p$ and $a - c_0 > b/q$ where $1 \leq k \leq p-1$ is the unique integer such that $qk+1 \in p\mathbb{Z}$. The sufficiency for the conditions that $\lfloor c/b \rfloor = p$ and $a - c_0 \leq b/q$ holds as $([c_0, a) + a\mathbb{Z}) \cap ([c_0, a) + \lambda + a\mathbb{Z}) = ([c_0, a) + a\mathbb{Z}) \cap ([c_0, a) - k(\lambda)b/q + a\mathbb{Z}) = \emptyset$ for all $\lambda \in [1, \lfloor c/b \rfloor]b \cap b\mathbb{Z}$, where $k(\lambda)$ is the unique integer in $[1, p-1]$ such that $k(\lambda)b/q + \lambda \in a\mathbb{Z}$. \square

5. PROPERTIES OF MAXIMAL INVARIANT SETS

To prove Theorems 2.4 and 2.5, we need some deep properties of the maximal invariant sets $\mathcal{S}_{a,b,c}$. Let us start from some examples of maximal invariant sets $\mathcal{S}_{a,b,c}$.

Example 5.1. Take $a = \pi/4, b = 1$ and $c = 23 - 11\pi/2$. The black holes of the corresponding transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ are $[17 - 21\pi/4, 18 - 11\pi/2) + \pi\mathbb{Z}/4$ and $[5 - 3\pi/2, 6 - 7\pi/4) + \pi\mathbb{Z}/4$ respectively, which can be transformed back and forth via the middle hole $[11 - 7\pi/2, 12 - 15\pi/4) + \pi\mathbb{Z}/4$; i.e.,

$$\begin{cases} R_{a,b,c}([5 - 3\pi/2, 6 - 7\pi/4) + \pi\mathbb{Z}/4) = [11 - 7\pi/2, 12 - 15\pi/4) + \pi\mathbb{Z}/4 \\ (R_{a,b,c})^2([5 - 3\pi/2, 6 - 7\pi/4) + \pi\mathbb{Z}/4) = [17 - 21\pi/4, 18 - 11\pi/2) + \pi\mathbb{Z}/4. \end{cases}$$

So the maximal invariant set

$$\begin{aligned} \mathcal{S}_{a,b,c} &= [18 - 23\pi/4, 11 - 7\pi/2) \cup [12 - 15\pi/4, 5 - 3\pi/2) \\ &\quad \cup [6 - 7\pi/4, 17 - 21\pi/4) + \pi\mathbb{Z}/4 \\ &\approx [-0.0642, 0.0044) \cup [0.2190, 0.2876) \cup [0.5022, 0.5066) + 0.7864\mathbb{Z} \end{aligned}$$

consists of intervals of different lengths on one period and contains a small neighborhood of the lattice $\pi\mathbb{Z}/4$, c.f. Figure 2.

Example 5.2. For the triple $(a, b, c) = (13/17, 1, 77/17)$, the black holes of the corresponding transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ are $[5/17, 9/17) + 13\mathbb{Z}/17$ and $[3/17, 7/17) + 13\mathbb{Z}/17$. Applying the transformation $R_{a,b,c}$ to the black hole of the transformation $\tilde{R}_{a,b,c}$, we obtain that

$$(5.1) \quad \begin{cases} R_{a,b,c}([3, 7)/17 + 13\mathbb{Z}/17) = ([5, 7) \cup [10, 12))/17 + 13\mathbb{Z}/17, \\ (R_{a,b,c})^2([3, 7)/17 + 13\mathbb{Z}/17) = ([5, 7) \cup [0, 2))/17 + 13\mathbb{Z}/17, \\ (R_{a,b,c})^3([3, 7)/17 + 13\mathbb{Z}/17) = [5, 9)/17 + 13\mathbb{Z}/17. \end{cases}$$

Thus the maximal invariant set

$$\begin{aligned} \mathcal{S}_{a,b,c} &= ([2, 3) \cup [9, 10) \cup [12, 13))/17 + 13\mathbb{Z}/17 \\ &\approx [0.1176, 0.1764) \cup [0.5294, 0.5882) \cup [0.7059, 0.7647) + 0.7647\mathbb{Z} \end{aligned}$$

consists of intervals of same length $1/17$ on the period $[0, 13/17)$ and contains small left neighborhood of the lattice $13\mathbb{Z}/17$, and its complement $\mathbb{R} \setminus \mathcal{S}_{a,b,c} = ([0, 2) \cup [3, 9) \cup [10, 12))/17 + 13\mathbb{Z}/17$ contains one big gap of size $6/17$, two small gaps of size $2/17$ on the period $[0, 13/17)$, and a small gap attached to the right-hand side of the lattice $13\mathbb{Z}/17$.

For the triple $(a, b, c) = (13/17, 1, 73/17)$, the maximal invariant set $\mathcal{S}_{a,b,c} = ([0, 1) \cup [7, 8) \cup [10, 11))/17 + 13\mathbb{Z}/17$ contains a small right neighborhood of the lattice $13\mathbb{Z}/17$, while its complement $\mathbb{R} \setminus \mathcal{S}_{a,b,c} =$

$([1, 7) \cup [8, 10) \cup [11, 13))/17 + 13\mathbb{Z}/17$ contains a small gap attached to the left-hand side of the lattice $13/17$.

For the triple $(a, b, c) = (13/17, 1, 75/17)$, the maximal invariant set

$$\begin{aligned}\mathcal{S}_{a,b,c} &= ([0, 3) \cup [7, 10) \cup [10, 13))/17 + 13\mathbb{Z}/17 \\ &= [0, 0.1765) \cup [0.4118, 0.5882) \cup [0.5882, 0.7647) + 0.7647\mathbb{Z}\end{aligned}$$

consists of intervals of “same” length $3/17$ and contains small left and right neighborhoods of the lattice $13\mathbb{Z}/17$. On the other hand, its complement $\mathbb{R} \setminus \mathcal{S}_{a,b,c} = [3, 7)/17 + 13\mathbb{Z}/17$ contains one big gap of size $4/17$ and two small gaps of size “zero” at $\{0, 10/17\}$ on the period $[0, 13/17)$, c.f. Figure 2.

Example 5.3. For the triple $(a, b, c) = (6/7, 1, 23/7)$, black holes of the corresponding transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ are $[1, 2)/7 + 6\mathbb{Z}/7$ and $[3, 4)/7 + 6\mathbb{Z}/7$ respectively. Observe that

$$\begin{cases} R_{a,b,c}([3, 4)/7 + 6\mathbb{Z}/7) = [0, 1)/7 + 6\mathbb{Z}/7 \\ (R_{a,b,c})^2([3, 4)/7 + 6\mathbb{Z}/7) = [4, 5)/7 + 6\mathbb{Z}/7 \\ (R_{a,b,c})^3([3, 4)/7 + 6\mathbb{Z}/7) = [1, 2)/7 + 6\mathbb{Z}/7, \end{cases}$$

which also implies that $R_{a,b,c}([3, 5)/7 + 6\mathbb{Z}/7) = [0, 2)/7 + 6\mathbb{Z}/7$. Therefore the maximal invariant set

$$\begin{aligned}\mathcal{S}_{a,b,c} &= [2, 3)/7 \cup [5, 6)/7 + 6\mathbb{Z}/7 \\ &= [0.2857, 0.4286) \cup [0.7143, 0.8571) + 0.8571\mathbb{Z}\end{aligned}$$

consists of intervals of length $1/7$, while its complement $\mathbb{R} \setminus \mathcal{S}_{a,b,c} = ([0, 2) \cup [3, 5))/7 + 6\mathbb{Z}/7$ consists of gaps of length $2/7$, c.f. Figure 2.

In the above examples, we see that the black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$ of the transformation $R_{a,b,c}$ attracts the black hole $[c - c_0, c - c_0 + b - a) + a\mathbb{Z}$ of the other transformation $\tilde{R}_{a,b,c}$ when applying $\mathcal{R}_{a,b,c}$ *finitely many times*, i.e.,

$$(R_{a,b,c})^L([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) = [c_0 + a - b, c_0) + a\mathbb{Z}$$

for some nonnegative integer L , and that holes

$$(R_{a,b,c})^l([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) = [c_0 + a - b, c_0) + a\mathbb{Z}, 0 \leq l \leq L - 1,$$

may or may not overlap with the black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$ of the transformation $R_{a,b,c}$, which depends on the ratio a/b being irrational or not.

For the first case that the ratio between time-spacing parameter a and frequency-spacing parameter b is irrational (i.e. $a/b \notin \mathbb{Q}$), we show that if $\mathcal{S}_{a,b,c} \neq \emptyset$ then the black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$ of the transformation $R_{a,b,c}$ and the black hole $[c - c_0, c - c_0 + b - a) + a\mathbb{Z}$ of the

transformation $\tilde{R}_{a,b,c}$ are inter-transformable through mutually disjoint periodic holes $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a] + a\mathbb{Z}) = (\tilde{R}_{a,b,c})^{D-n}([c_0 + a - b, c_0] + a\mathbb{Z})$, $0 \leq n \leq D$, where D is a nonnegative integer. Furthermore the complement of the set $\mathcal{S}_{a,b,c}$ is the union of mutually disjoint holes of *same size*, but the set $\mathcal{S}_{a,b,c}$ is the union of disjoint intervals of “different” sizes.

Theorem 5.4. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, $\lfloor c/b \rfloor \geq 2$ and $a/b \notin \mathbb{Q}$. Assume that $\mathcal{S}_{a,b,c} \neq \emptyset$. Then there exists a nonnegative integer $D \leq \lfloor a/(b - a) \rfloor - 1$ such that*

$$(5.2) \quad (\tilde{R}_{a,b,c})^n(c_0 + a - b) + a\mathbb{Z} = (R_{a,b,c})^{D-n}(c - c_0) + a\mathbb{Z}$$

for all $0 \leq n \leq D$,

$$(5.3) \quad ((R_{a,b,c})^n(c - c_0) + [0, b - a] + a\mathbb{Z}) \cap ((R_{a,b,c})^{n'}(c - c_0) + [0, b - a] + a\mathbb{Z}) = \emptyset$$

for all $0 \leq n \neq n' \leq D$. Moreover

$$(5.4) \quad \begin{aligned} \mathbb{R} \setminus \mathcal{S}_{a,b,c} &= \bigcup_{n=0}^D ((R_{a,b,c})^n(c - c_0) + [0, b - a] + a\mathbb{Z}) \\ &= \bigcup_{n=0}^D (R_{a,b,c})^n([c - c_0, c - c_0 + b - a] + a\mathbb{Z}) \\ &= \bigcup_{n=0}^D (\tilde{R}_{a,b,c})^n([c_0 + a - b, c_0] + a\mathbb{Z}) \\ &= \bigcup_{n=0}^D ((\tilde{R}_{a,b,c})^n(c_0 + a - b) + [0, b - a] + a\mathbb{Z}). \end{aligned}$$

For the second case that a/b is rational, we write $a/b = p/q$ for some coprime integers p and q . We restrict ourselves to consider $c \in b\mathbb{Z}/q$ because for $c \notin b\mathbb{Z}/q$, $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if both $\mathcal{G}(\chi_{[0, \lfloor qc/b \rfloor b/q]}, a\mathbb{Z} \times \mathbb{Z}/b)$ and $\mathcal{G}(\chi_{[0, \lfloor qc/b+1 \rfloor b/q]}, a\mathbb{Z} \times \mathbb{Z}/b)$ are Gabor frames, see [25, Section 3.3.6.1] and the conclusion (XIV) of Theorem 2.5. Observe that for $a/b = p/q$ and $c \in b\mathbb{Z}/q$,

$$(5.5) \quad \mathbf{M}_{a,b,c}(t) = \mathbf{M}_{a,b,c}(\lfloor qt/b \rfloor b/q), \quad t \in \mathbb{R},$$

which implies that

$$(5.6) \quad \mathcal{D}_{a,b,c} = \mathcal{D}_{a,b,c} \cap b\mathbb{Z}/q + [0, b/q] \quad \text{and} \quad \mathcal{S}_{a,b,c} = \mathcal{S}_{a,b,c} \cap b\mathbb{Z}/q + [0, b/q].$$

Even further, the sets $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$ are essentially *finite* sets, as they are completely determined by their restrictions to the finite set $\{0, b/q, \dots, (p-1)b/q\}$,

$$(5.7) \quad \begin{cases} \mathcal{D}_{a,b,c} = \mathcal{D}_{a,b,c} \cap \{0, b/q, \dots, (p-1)b/q\} + pb\mathbb{Z}/q + [0, b/q] \\ \mathcal{S}_{a,b,c} = \mathcal{S}_{a,b,c} \cap \{0, b/q, \dots, (p-1)b/q\} + pb\mathbb{Z}/q + [0, b/q], \end{cases}$$

by (5.6) and the periodic property and (3.3). In the next theorem, we show that the maximal invariant set $\mathcal{S}_{a,b,c}$ is the union of half-open intervals of *same size* and its complement $\mathbb{R} \setminus \mathcal{S}_{a,b,c}$ is the union of mutually disjoint gaps of “*two different*” sizes.

Theorem 5.5. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c, b - a < c_0 := c - \lfloor c/b \rfloor b < a, \lfloor c/b \rfloor \geq 2, a/b = p/q$ for some coprime integers p and q , and $c/b \in \mathbb{Z}/q$. Assume that $\mathcal{S}_{a,b,c} \neq \emptyset$. Then there are $\delta \in [0, c_0 + a - b] \cap b\mathbb{Z}/q, \delta' \in [c_0 - a, 0] \cap b\mathbb{Z}/q$, and nonnegative integers N_1 and N_2 with the following properties:*

(i) *At least one of δ and δ' is equal to zero; i.e.,*

$$(5.8) \quad \delta\delta' = 0.$$

(ii) *The periodic gaps $(R_{a,b,c})^n(c - c_0 + [\delta', b - a + \delta]) + a\mathbb{Z}, 0 \leq n \leq N_1$, have length $b - a + \delta - \delta'$, and the periodic gap $(R_{a,b,c})^{N_1}(c - c_0 + [\delta', b - a + \delta]) + a\mathbb{Z}$ coincides with $[c_0 + a - b - \delta, c_0 - \delta'] + a\mathbb{Z}$ that contains the black hole of the piecewise linear transformation $R_{a,b,c}$; i.e.,*

$$(5.9) \quad \begin{aligned} & (R_{a,b,c})^n(c - c_0 + [\delta', b - a + \delta]) + a\mathbb{Z} \\ = & (R_{a,b,c})^n(c - c_0 + b - a + \delta) + [a - b + \delta' - \delta, 0] + a\mathbb{Z} \end{aligned}$$

for all $0 \leq n \leq N_1$, and

$$(5.10) \quad (R_{a,b,c})^{N_1}(c - c_0 + b - a + \delta) + a\mathbb{Z} = c_0 - \delta' + a\mathbb{Z}.$$

(iii) *The periodic gaps $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 - \delta'] \setminus [c_0 + a - b, c_0]) + a\mathbb{Z}, 1 \leq m \leq N_2$, have length $\delta - \delta'$ and the periodic gap $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 - \delta'] \setminus [c_0 + a - b, c_0]) + a\mathbb{Z}$ with $m = N_2$ is the same as $[\delta', \delta] + a\mathbb{Z}$, provided that $\delta - \delta' \neq 0$; i.e.,*

$$(5.11) \quad \begin{aligned} & (R_{a,b,c})^m([c_0 + a - b - \delta, c_0 - \delta'] \setminus [c_0 + a - b, c_0]) + a\mathbb{Z} \\ = & (R_{a,b,c})^m(c_0 - \delta') + [\delta' - \delta, 0] + a\mathbb{Z} \end{aligned}$$

for all $1 \leq m \leq N_2$, and

$$(5.12) \quad (R_{a,b,c})^{N_2}([c_0 + a - b - \delta, c_0 - \delta'] \setminus [c_0 + a - b, c_0]) + a\mathbb{Z} = [\delta', \delta] + a\mathbb{Z}.$$

(iii)' $(R_{a,b,c})^{N_2}(c_0) \in a\mathbb{Z}$ provided that $\delta = \delta' = 0$.

(iv) *The periodic gaps $(R_{a,b,c})^n(c - c_0 + [\delta', b - a + \delta]) + a\mathbb{Z}, 0 \leq n \leq N_1$, of length $b - a + \delta - \delta'$, and $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 - \delta'] \setminus [c_0 + a - b, c_0]) + a\mathbb{Z}, 1 \leq m \leq N_2$, of length $\delta - \delta'$ together with neighboring intervals of length $b/(2q)$ at each side are mutually disjoint, provided that $\delta - \delta' \neq 0$.*

(iv)' The periodic gaps $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a]) + a\mathbb{Z}$, $0 \leq n \leq N_1$, of length $b - a$ associated with neighboring intervals of length $b/(2q)$ at each side, and $(R_{a,b,c})^m(c_0) + [-b/(2q), b/(2q)] + a\mathbb{Z}$, $1 \leq m \leq N_2$, are mutually disjoint, provided that $\delta = \delta' = 0$.

(v) The complement of the set $\mathcal{S}_{a,b,c}$ is the union of periodic gaps $(R_{a,b,c})^n(c - c_0 + [\delta', b - a + \delta]) + a\mathbb{Z}$, $0 \leq n \leq N_1$, and $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 - \delta']) + a\mathbb{Z}$, $1 \leq m \leq N_2$; i.e.,

$$\begin{aligned} \mathbb{R} \setminus \mathcal{S}_{a,b,c} &= \left(\bigcup_{n=0}^{N_1} (R_{a,b,c})^n(c - c_0 + [\delta', b - a + \delta]) + a\mathbb{Z} \right) \\ &\quad \cup \left(\bigcup_{m=1}^{N_2} (R_{a,b,c})^m[c_0 + a - b - \delta, c_0 - \delta'] + a\mathbb{Z} \right) \\ (5.13) \quad &= \bigcup_{n=0}^{N_1+N_2} (R_{a,b,c})^n(c - c_0 + [\delta', b - a + \delta]) + a\mathbb{Z}. \end{aligned}$$

(vi) The set $\mathcal{S}_{a,b,c}$ is composed of intervals of same length,

$$(5.14) \quad \mathcal{S}_{a,b,c} = \bigcup_{n=0}^{N_1+N_2} (R_{a,b,c})^n(c - c_0 + b - a + \delta) + [0, h] + a\mathbb{Z}$$

where

$$(5.15) \quad (N_1 + N_2 + 1)(h + \delta - \delta') + (N_1 + 1)(b - a) = a.$$

We remark that for triples $(13/17, 1, 77/17)$, $(13/17, 1, 73/17)$ and $(13/17, 1, 75/77)$ in Example 5.2 and $(6/7, 1, 23/7)$ in Example 5.3, the corresponding pair (δ, δ') in Theorem 5.5 is given by $(2/17, 0)$, $(0, -2/17)$, $(0, 0)$, $(1/7, 0)$ respectively.

Listed below are some comparisons of the maximal invariant set $\mathcal{S}_{a,b,c}$ for the first case that $a/b \notin \mathbb{Q}$ and for the second case that $a/b = p/q$ for some coprime integers p and q and $c \in b\mathbb{Z}/q$, where δ, δ' are as in Theorem 5.5, c.f. Examples 5.1, 5.2 and 5.3, and Figure 2:

- For the second case, the restriction of the maximal invariant periodic set $\mathcal{S}_{a,b,c}$ on the period $[0, a)$ is the union of finitely many half-open intervals of same size if $\delta - \delta' > 0$, while for the first case it contains the union of finitely many half-open intervals of different sizes.
- For the second case, the restrictions of the periodic complement $\mathbb{R} \setminus \mathcal{S}_{a,b,c}$ on the period $[0, a)$ consists of finitely many mutually disjoint periodic gaps of lengths $b - a + \delta - \delta'$ and $\delta - \delta'$ if $\delta - \delta' > 0$, while for the first case it includes finitely many mutually disjoint periodic holes of same size $b - a$.
- For the second case, either $[0, \delta)$ or $[\delta', 0)$ is a gap contained in the periodic complement $\mathbb{R} \setminus \mathcal{S}_{a,b,c}$ if $\delta - \delta' > 0$, while for the first case no left or right holes at the origin are contained in the periodic complement $\mathbb{R} \setminus \mathcal{S}_{a,b,c}$, see Lemma 3.11.

- The complement of the maximal invariant set $\mathcal{S}_{a,b,c}$ for the second case with $\delta = \delta' = 0$ looks “similar” to the one for the first case, since their restrictions on one period both consist of finitely many mutually disjoint half-open intervals of length $b-a$. On the other hand, they are “different” if we adhere intervals of sufficiently small length ϵ to each side of those half-open intervals. For the second case, those gaps with neighboring intervals has the following “loop” structure via the piecewise linear transformation $R_{a,b,c}$ with shrinking and expanding at its black hole and the origin:

$$\begin{array}{ccc}
 \boxed{c - c_0 + [-\epsilon, \epsilon + b - a) + a\mathbb{Z}} & \longleftarrow & \boxed{[-\epsilon, \epsilon) + a\mathbb{Z}} \\
 \downarrow & & \uparrow \\
 \vdots & & \vdots \\
 \downarrow & & \uparrow \\
 \boxed{[c_0 + a - b, c_0) + [-\epsilon, \epsilon) + a\mathbb{Z}} & \longrightarrow & \boxed{R_{a,b,c}(c_0) + [-\epsilon, \epsilon) + a\mathbb{Z}}
 \end{array}$$

Hence the piecewise linear transformation $R_{a,b,c}$ has “finite” order, as there exists a positive integer L such that $(R_{a,b,c})^L$ is an “identity” operator I in the sense that $(R_{a,b,c})^L(t) + a\mathbb{Z} = t + a\mathbb{Z}$ for all $t \in \mathcal{S}_{a,b,c}$, see Theorem 5.8. But for the first case, applying the transform $R_{a,b,c}$ finitely many time, the black hole of the piecewise linear transformation $\tilde{R}_{a,b,c}$ is “totally” attracted by the black hole of the piecewise linear transformation $R_{a,b,c}$, which indicates that the piecewise linear transformation $R_{a,b,c}$ has “infinite” order.

- The complement of the maximal invariant set $\mathcal{S}_{a,b,c}$ for the case that $N_2 = 0$, c.f. Example 5.3, looks also “similar” to the one for the first case, in the sense that their restrictions on one period both consist of finitely many mutually disjoint half-open intervals of same length, but the lengths are different for those two cases. Also for the case that $N_2 = 0$, the maximal invariant set $\mathcal{S}_{a,b,c}$ is the union of finitely many half-open intervals of same size.
- For both cases, the black hole of the piecewise linear transformation $R_{a,b,c}$ attracts the black hole of the piecewise linear transformation $\tilde{R}_{a,b,c}$ when applying the piecewise linear transformation $R_{a,b,c}$ finitely many times,

$$[c_1, c_1 + b - a) + a\mathbb{Z} \xrightarrow{R_{a,b,c}} \cdots \xrightarrow{R_{a,b,c}} [c_0 + a - b, c_0) + a\mathbb{Z}$$

see (5.36), (5.61) and (5.68). Moreover,

$$\begin{aligned} \mathcal{S}_{a,b,c} &= \mathbb{R} \setminus \left(\bigcup_{n=0}^{\infty} (R_{a,b,c})^n([c_1, c_1 + b - a] + a\mathbb{Z}) \right) \\ (5.16) \quad &= \mathbb{R} \setminus \left(\bigcup_{n=0}^L (R_{a,b,c})^n([c_1, c_1 + b - a] + a\mathbb{Z}) \right) \end{aligned}$$

for some nonnegative integer $L \geq 0$ when $\mathcal{S}_{a,b,c} \neq \emptyset$.

Having explicit construction of the maximal invariant set $\mathcal{S}_{a,b,c}$ in Theorems 5.4 and 5.5, we next consider covering property of the maximal invariant set $\mathcal{S}_{a,b,c}$ and then its application to characterize (2.11).

Theorem 5.6. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, $\lfloor c/b \rfloor \geq 2$, $b - a < c_0 := c - \lfloor c/b \rfloor b < a$ and $0 \leq c_1 := c - c_0 - \lfloor (c - c_0)/a \rfloor a \leq 2a - b$, and let $\mathcal{S}_{a,b,c}$ be as in (2.9). If $\mathcal{S}_{a,b,c} \neq \emptyset$ and either $a/b \notin \mathbb{Q}$ or $a/b = p/q$ and $c \in b\mathbb{Z}/q$ for some coprime integers p and q , then*

$$(5.17) \quad (\mathcal{S}_{a,b,c} \cap ([0, c_0 + a - b] + a\mathbb{Z}) + \lfloor c/b \rfloor b) \cup \left(\bigcup_{k=0}^{\lfloor c/b \rfloor - 1} (\mathcal{S}_{a,b,c} + kb) \right) = \mathbb{R}.$$

As a direct application of the above theorem, we have that

$$\bigcup_{k=0}^{\lfloor c/b \rfloor - 1} (\mathcal{S}_{a,b,c} + kb) \subset \mathbb{R} \subset \bigcup_{k=0}^{\lfloor c/b \rfloor} (\mathcal{S}_{a,b,c} + kb).$$

So roughly speaking, the maximal invariant periodic set $\mathcal{S}_{a,b,c}$ is either an empty set or its $(\lfloor c/b \rfloor + 1)$ copies would cover the whole line. Recall that the set $\mathcal{D}_{a,b,c}$ can be obtained from the maximal invariant set $\mathcal{S}_{a,b,c}$ by some set operations (Theorem 4.2). This together with Theorem 5.6 leads to the following equivalence between the empty set property for $\mathcal{D}_{a,b,c}$ and an equality about the Lebesgue measure of the maximal invariant set $\mathcal{S}_{a,b,c}$.

Theorem 5.7. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, $0 < c_1 := \lfloor c/b \rfloor b - \lfloor (\lfloor c/b \rfloor b / a) \rfloor a < 2a - b$ and $\lfloor c/b \rfloor \geq 2$. Assume that $\mathcal{S}_{a,b,c} \neq \emptyset$ and either $a/b \notin \mathbb{Q}$ or $a/b = p/q$ and $c/b \in \mathbb{Z}/q$ for some coprime integers p and q . Then $\mathcal{D}_{a,b,c} = \emptyset$ if and only if*

$$(5.18) \quad (\lfloor c/b \rfloor + 1)|\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b]| + \lfloor c/b \rfloor |\mathcal{S}_{a,b,c} \cap [c_0, a]| = a.$$

For the triple $(a, b, c) = (\pi/4, 1, 23 - 11\pi/2)$ in Example 5.1, the equality (5.18) holds, and hence $\mathcal{G}(\chi_{[0, 23 - 11\pi/2)}, \pi\mathbb{Z}/4 \times \mathbb{Z})$ is a Gabor frame even though the maximal invariant set $\mathcal{S}_{a,b,c}$ is not an empty set. For the triple $(a, b, c) = (\sqrt{3}/2, 1, 15\sqrt{3}/2)$ with irrational ratio between a and b , $\mathcal{G}(\chi_{[0, 15\sqrt{3}/2)}, \sqrt{3}\mathbb{Z}/2 \times \mathbb{Z})$ is not a Gabor frame as (5.18) does not hold. In fact, in this case

$$\begin{aligned} \mathcal{D}_{a,b,c} = \mathcal{S}_{a,b,c} &= ([12 - 7\sqrt{3}, 7 - 4\sqrt{3}) \cup [8 - 9\sqrt{3}/2, 3 - 3\sqrt{3}/2) \\ &\quad \cup [4 - 2\sqrt{3}, 11 - 6\sqrt{3})) + \sqrt{3}\mathbb{Z}/2. \end{aligned}$$

For the triple $(a, b, c) = (6/7, 1, 23/7)$ in Example 5.3 and the triples $(a, b, c) = (13, 17, 77)/17$ and $(13, 17, 73)/17$ in Example 5.1, the equality (5.18) holds, but for the triple $(13, 17, 75)/17$ in Example 5.1 the equality (5.18) does not hold. Thus Gabor systems $\mathcal{G}(\chi_{[0, 23/7]}, 6\mathbb{Z}/7 \times \mathbb{Z})$, $\mathcal{G}(\chi_{[0, 77/17]}, 13\mathbb{Z}/17 \times \mathbb{Z})$ and $\mathcal{G}(\chi_{[0, 73/17]}, 13\mathbb{Z}/17 \times \mathbb{Z})$ are Gabor frames even though the maximal invariant sets are nontrivial, but the Gabor system $\mathcal{G}(\chi_{[0, 75/17]}, 13\mathbb{Z}/17 \times \mathbb{Z})$ is not a Gabor frame.

In this section, we finally consider the restriction of transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ on the maximal invariant set $\mathcal{S}_{a,b,c}$. The finite-interval property for the maximal invariant set $\mathcal{S}_{a,b,c}$ in Theorems 5.4 and 5.5 can also be interpreted as its complement consists of finitely many holes on a period. So we may shrink those holes into points, which maps the maximal invariant set $\mathcal{S}_{a,b,c}$ into the real line with marks. More importantly, after performing the above holes-removal surgery, the set $\mathcal{K}_{a,b,c}$ of marks on the line is a *finite cyclic group* if $a/b = p/q$ and $c \in b\mathbb{Z}/q$ for some coprime integers p and q , and the set $\mathcal{K}_{a,b,c}$ of marks on the line is a finite subset of infinite cyclic group if $a/b \notin \mathbb{Q}$, and the nonlinear application of the transformation $R_{a,b,c}$ on the maximal invariant set $\mathcal{S}_{a,b,c}$ becomes a *rotation* on the circle $\mathbb{R}/Y_{a,b,c}(a)\mathbb{Z}$ with marks $\mathcal{K}_{a,b,c}$, see Appendix C for non-ergodicity of the piecewise linear transformation $R_{a,b,c}$.

Theorem 5.8. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, $\lfloor c/b \rfloor \geq 2$, $b - a < c_0 := c - \lfloor c/b \rfloor b < a$ and $0 \leq c_1 := c - c_0 - \lfloor (c - c_0)/a \rfloor a \leq 2a - b$, and let $\mathcal{S}_{a,b,c}$ and $Y_{a,b,c}$ be as in (2.9) and (2.16) respectively. Then the following statements hold.*

- (i) *If $\mathcal{S}_{a,b,c} \neq \emptyset$ and either $a/b \notin \mathbb{Q}$ or $a/b = p/q$ and $c \in b\mathbb{Z}/q$ for some coprime integers p and q , then under the isomorphism $Y_{a,b,c}$ from $\mathcal{S}_{a,b,c}$ to the line with marks, the action to apply the piecewise linear transformation $R_{a,b,c}$ on $\mathcal{S}_{a,b,c}$ becomes a shift on the line with marks; i.e.,*

(5.19)

$$Y_{a,b,c}(R_{a,b,c}(t) + a\mathbb{Z}) = Y_{a,b,c}(t) + Y_{a,b,c}(c_1 + b - a) + Y_{a,b,c}(a)\mathbb{Z} \quad \text{for all } t \in \mathcal{S}_{a,b,c}.$$

- (ii) *If $\mathcal{S}_{a,b,c} \neq \emptyset$, $a/b = p/q$ and $c \in b\mathbb{Z}/q$ for some coprime integers p and q , then marks on the line (i.e., images of the gaps in the complement of the set $\mathcal{S}_{a,b,c}$ under the isomorphism $Y_{a,b,c}$) form a finite cyclic group generated by $Y_{a,b,c}(c_1 + b - a) + Y_{a,b,c}(a)\mathbb{Z}$,*

$$\mathcal{K}_{a,b,c} = Y_{a,b,c}(c_1 + b - a)\mathbb{Z} + Y_{a,b,c}(a)\mathbb{Z} = \gcd(Y_{a,b,c}(c_1 + b - a), Y_{a,b,c}(a))\mathbb{Z},$$

where $\mathcal{K}_{a,b,c}$ is the set of marks on the line.

- (iii) If $\mathcal{S}_{a,b,c} \neq \emptyset$ and $a/b \notin \mathbb{Q}$, the set $\mathcal{K}_{a,b,c}$ of marks on the line is given by

$$\mathcal{K}_{a,b,c} = \cup_{n=1}^M (nY_{a,b,c}(c_1 + b - a) + Y_{a,b,c}(a)\mathbb{Z})$$

where M is the unique positive integer such that $MY_{a,b,c}(c_1 + b - a) - Y_{a,b,c}(c_0) \in Y_{a,b,c}(a)$.

In the next four subsections, we prove Theorems 5.4, 5.5, 5.6 and 5.7, and 5.8 respectively.

5.1. Maximal invariant sets with irrational time-frequency lattices. To prove Theorem 5.4, we need a characterization of non-empty-set property for the maximal invariant set $\mathcal{S}_{a,b,c}$.

Lemma 5.9. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c, b - a < c_0 := c - \lfloor c/b \rfloor b < a, \lfloor c/b \rfloor \geq 2$ and $a/b \notin \mathbb{Q}$. Then $\mathcal{S}_{a,b,c} \neq \emptyset$ if and only if there exists a nonnegative integer $D \leq \lfloor a/(b - a) \rfloor - 1$ such that $(R_{a,b,c})^D([c - c_0, c + b - c_0 - a] + a\mathbb{Z}) = [c_0 + a - b, c_0] + a\mathbb{Z}$ and that $(R_{a,b,c})^n([c - c_0, c + b - c_0 - a] + a\mathbb{Z}), 0 \leq n \leq D - 1$, has their closures being mutually disjoint and contained in $(0, c_0 + a - b) \cup (c_0, a) + a\mathbb{Z}$.*

The sufficiency of the above lemma follows from the invariance of the set $\cup_{n=0}^D (R_{a,b,c})^n([c - c_0, c - c_0 + b - a] + a\mathbb{Z})$ under transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$, and the minimality of the set $\mathbb{R} \setminus \mathcal{S}_{a,b,c}$ in Theorem 4.1. To prove the necessity in Lemma 5.9, we let D be the minimal nonnegative integer such that $(R_{a,b,c})^D([c - c_0, c + b - c_0 - a] + a\mathbb{Z}) \cap ([c_0 + a - b, c_0] + a\mathbb{Z}) \neq \emptyset$. The existence of such an integer D follows from the mutually disjointness of holes $(R_{a,b,c})^n([c - c_0, c + b - c_0 - a] + a\mathbb{Z}), 0 \leq n \leq D - 1$ (and their closures), see (5.29). We then prove that $(R_{a,b,c})^D([c - c_0, c + b - c_0 - a] + a\mathbb{Z}) = [c_0 + a - b, c_0] + a\mathbb{Z}$ by applying Lemma 3.11 and Propositions 3.8 and 3.10, see (5.36). Now assuming that Lemma 5.9 holds, we start our proof of Theorem 5.4.

Proof of Theorem 5.4. Assume that $\mathcal{S}_{a,b,c} \neq \emptyset$. By Lemma 5.9, there exists a nonnegative integer $D \leq \lfloor a/(b - a) \rfloor - 1$ such that $(R_{a,b,c})^D([c - c_0, c + b - c_0 - a] + a\mathbb{Z}) = [c_0 + a - b, c_0] + a\mathbb{Z}$, and $(R_{a,b,c})^n([c - c_0, c + b - c_0 - a] + a\mathbb{Z}), 0 \leq n \leq D - 1$, have their closures being mutually disjoint and contained in $(0, c_0 + a - b) \cup (c_0, a) + a\mathbb{Z}$. Then $\cup_{n=0}^D (R_{a,b,c})^n([c - c_0, c - c_0 + b - a] + a\mathbb{Z})$ is minimal set that contains black holes of transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$, and that is invariant under those transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$. Here the invariance holds as (5.20)

$$(R_{a,b,c})^n([c - c_0, c - c_0 + b - a] + a\mathbb{Z}) = (\tilde{R}_{a,b,c})^{D-n}([c_0 + a - b, c_0] + a\mathbb{Z})$$

for all $0 \leq n \leq D$ by Proposition 3.7 and the property that

$$(5.21) \quad (R_{a,b,c})^n([c-c_0, c+b-c_0-a] + a\mathbb{Z}) \subset (0, c_0+a-b) \cup (c_0, a) + a\mathbb{Z}$$

for all $0 \leq n \leq D-1$. This together with Theorem 4.1 proves that

$$(5.22) \quad \mathcal{S}_{a,b,c} = \mathbb{R} \setminus \bigcup_{n=0}^D (R_{a,b,c})^n([c-c_0, c-c_0+b-a] + a\mathbb{Z}).$$

Hence the conclusions (5.2), (5.3) and (5.4) follow from (2.13), (2.14) and (5.20)–(5.22). \square

We finish this subsection with the proof of Lemma 5.9.

Proof of Lemma 5.9. (\Leftarrow) By (3.64), $|(R_{a,b,c})^n([c-c_0, c-c_0+b-a] + a\mathbb{Z}) \cap [0, a]| \leq b-a$ for all $0 \leq n \leq D$, which together with the periodicity (2.13) of the transformation $R_{a,b,c}$ implies that

$$|A_D \cap [0, a]| \leq (D+1)(b-a) < a,$$

where $A_D = \bigcup_{n=0}^D (R_{a,b,c})^n([c-c_0, c-c_0+b-a] + a\mathbb{Z})$. Hence

$$(5.23) \quad A_D \neq \mathbb{R}.$$

By (5.23), the sufficiency reduces to proving that

$$(5.24) \quad \mathbb{R} \setminus A_D \subset \mathcal{S}_{a,b,c}.$$

(In particular, $\mathbb{R} \setminus A_D = \mathcal{S}_{a,b,c}$ by Theorem 4.1 and Proposition 4.3). Recall that the black hole $[c-c_0, c-c_0+b-a] + a\mathbb{Z}$ of the transformation $\tilde{R}_{a,b,c}$ is contained in A_D , and that the black hole $[c_0+a-b, c_0] + a\mathbb{Z}$ of the transformation $R_{a,b,c}$ is also contained in A_D as it is same as $(R_{a,b,c})^D([c-c_0, c-c_0+b-a] + a\mathbb{Z})$ by the assumption. Then we obtain from (3.48) and (3.50) that

$$(5.25) \quad \tilde{R}_{a,b,c}(\mathbb{R} \setminus A_D) \subset \mathbb{R} \setminus A_D.$$

Similarly we get from (2.13) and (3.50) that $R_{a,b,c}(t) \notin [c-c_0, c-c_0+b-a] + a\mathbb{Z}$ and that $R_{a,b,c}(t) \notin \bigcup_{n=1}^D (R_{a,b,c})^n([c-c_0, c-c_0+b-a] + a\mathbb{Z})$ for any $t \notin A_D$. This implies that

$$(5.26) \quad R_{a,b,c}(\mathbb{R} \setminus A_D) \subset \mathbb{R} \setminus A_D.$$

Take any $t \in \mathbb{R} \setminus A_D$, then $(R_{a,b,c})^n(t)$ and $(\tilde{R}_{a,b,c})^n(t)$, $n \geq 0$, belong to $\mathbb{R} \setminus A_D$ by (5.25) and (5.26) and hence they do not fall in the black holes of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$. Therefore $t \in \mathcal{S}_{a,b,c}$ by Proposition 4.3, and hence (5.24) is established.

(\Rightarrow) First we need the following claim:

Claim 5.10. *There exists a nonnegative integer $n \leq \lfloor a/(b-a) \rfloor - 1$ such that*

$$(5.27) \quad ((R_{a,b,c})^n([c-c_0, c-c_0+b-a] + a\mathbb{Z})) \cap ([c_0+a-b, c_0] + a\mathbb{Z}) \neq \emptyset.$$

Proof. Suppose, on the contrary, that

$$(5.28) \quad ((R_{a,b,c})^n([c-c_0, c-c_0+b-a]+a\mathbb{Z})) \cap ([c_0+a-b, c_0]+a\mathbb{Z}) = \emptyset$$

for all $0 \leq n \leq \lfloor a/(b-a) \rfloor - 1$. We first need to prove that

$$(5.29) \quad (R_{a,b,c})^n([c-c_0, c-c_0+b-a]+a\mathbb{Z})) \cap (R_{a,b,c})^{n'}([c-c_0, c-c_0+b-a]+a\mathbb{Z})) = \emptyset$$

for all $0 \leq n \neq n' \leq \lfloor a/(b-a) \rfloor - 1$. Suppose on the contrary that (5.29) does not hold. Then there exists $t \in ((R_{a,b,c})^n([c-c_0, c-c_0+b-a]+a\mathbb{Z})) \cap ((R_{a,b,c})^{n'}([c-c_0, c-c_0+b-a]+a\mathbb{Z}))$. By (3.50) and (5.28), $(\hat{R}_{a,b,c})^{\max(n,n')-1}(t) \in (R_{a,b,c}([c-c_0, c-c_0+b-a]+a\mathbb{Z})) \cap ([c-c_0, c-c_0+b-a]+a\mathbb{Z})$, which is a contradiction as $(R_{a,b,c}([c-c_0, c-c_0+b-a]+a\mathbb{Z})) \cap ([c-c_0, c-c_0+b-a]+a\mathbb{Z}) \subset ([c-c_0+b, c-c_0+2a]+a\mathbb{Z}) \cap ([c-c_0, c-c_0+b-a]+a\mathbb{Z}) = \emptyset$ by (2.13) and (5.28). Hence (5.29) is established.

By (3.65) and (5.28), we have that

$$(5.30) \quad |((R_{a,b,c})^n([c-c_0, c-c_0+b-a]+a\mathbb{Z})) \cap [0, a]| = b-a$$

for all $0 \leq n \leq \lfloor a/(b-a) \rfloor - 1$. This together with (5.29) implies that

$$\begin{aligned} a &\geq \left| \bigcup_{n=0}^{\lfloor a/(b-a) \rfloor - 1} ((R_{a,b,c})^n([c-c_0, c-c_0+b-a]+a\mathbb{Z})) \cap [0, a] \right| \\ &\quad + |[c_0+a-b, c_0]| \\ &\geq \sum_{n=0}^{\lfloor a/(b-a) \rfloor - 1} |((R_{a,b,c})^n([c-c_0, c-c_0+b-a]+a\mathbb{Z})) \cap [0, a]| \\ &\quad + (b-a) \\ &= (\lfloor a/(b-a) \rfloor + 1)(b-a) > a, \end{aligned}$$

which is a contradiction. This completes the proof of Claim 5.10. \square

We return to work on the proof of the sufficiency. Let D be the smallest nonnegative integer such that

$$(5.31) \quad ((R_{a,b,c})^D([c-c_0, c-c_0+b-a]+a\mathbb{Z})) \cap ([c_0+a-b, c_0]+a\mathbb{Z}) \neq \emptyset.$$

Then $D \leq \lfloor a/(b-a) \rfloor - 1$ by (5.27). By the definition of the integer D , we have that

$$(5.32) \quad ((R_{a,b,c})^n([c-c_0, c-c_0+b-a]+a\mathbb{Z})) \cap ([c_0+a-b, c_0]+a\mathbb{Z}) = \emptyset$$

for all $0 \leq n < D-1$. Following the argument used to establish (5.29), we have that

$$(5.33) \quad \left((R_{a,b,c})^n([c-c_0, c-c_0+b-a]+a\mathbb{Z}) \right) \cap \left((R_{a,b,c})^{n'}([c-c_0, c-c_0+b-a]+a\mathbb{Z}) \right) = \emptyset$$

for all $0 \leq n \neq n' \leq D-1$. We claim that for all $0 \leq n < D-1$,

$$(5.34) \quad (R_{a,b,c})^n([c-c_0, c-c_0+b-a] + a\mathbb{Z}) = [\tilde{u}_n, u_n] + a\mathbb{Z}$$

with

$$(5.35) \quad u_n = \tilde{u}_n + b - a \quad \text{and} \quad [\tilde{u}_n, u_n] \subset (0, c_0 + a - b) \text{ or } (c_0, a);$$

that is, $(R_{a,b,c})^n([c-c_0, c-c_0+b-a] + a\mathbb{Z})$ are periodic holes of length $b-a$ ‘strictly’ contained in $(0, c_0 + a - b) \cup (c_0, a) + a\mathbb{Z}$. Let u_0 be the unique number in $(0, a]$ such that $c-c_0+b-a-u_0 \in a\mathbb{Z}$, and $\tilde{u}_0 < u_0$ be the largest number such that $c-c_0-\tilde{u}_0 \in a\mathbb{Z}$. Then $[c-c_0, c-c_0+b-a] + a\mathbb{Z} = [\tilde{u}_0, u_0] + a\mathbb{Z}$. Furthermore $\tilde{u}_0 > 0$ and $u_0 < a$ as otherwise either $((0, \epsilon) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset$ or $((-\epsilon, 0) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset$ for some small $\epsilon > 0$ by Proposition 3.10, which contradicts to the conclusion (3.55) in Lemma 3.11. This together with (5.32) and the definitions of \tilde{u}_0 and u_0 implies that $u_0 = \tilde{u}_0 + b - a$ and $[\tilde{u}_0, u_0] \subset (0, c_0 + a - b]$ or $[c_0, a)$. Therefore it suffices to prove $u_0 \neq c_0 + a - b$ and $\tilde{u}_0 \neq c_0$. Suppose on the contrary that $u_0 = c_0 + a - b$, then $(R_{a,b,c}([0, \epsilon] + a\mathbb{Z})) \cap \mathcal{S}_{a,b,c} = ([u_0, u_0 + \epsilon] + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} \subset ([c_0 + a - b, c_0] + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset$ for sufficiently small $\epsilon \in (0, c_0 + a - b)$ by (3.54). Thus $([0, \epsilon] + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset$ by (3.29) and Proposition 3.8, which contradicts to (3.55) in Lemma 3.11. This proves that $u_0 \neq c_0 + a - b$. Similarly we can prove that $\tilde{u}_0 \neq c_0$. Therefore the conclusions (5.34) and (5.35) hold for $n = 0$. Inductively, we assume that $(R_{a,b,c})^n([c-c_0, c-c_0+b-a] + a\mathbb{Z}) = [\tilde{u}_n, u_n] + a\mathbb{Z}$ with $u_n = \tilde{u}_n + b - a$ and $[\tilde{u}_n, u_n] \subset (0, c_0 + a - b)$ or (c_0, a) . Then we see that $(R_{a,b,c})^{n+1}([c-c_0, c-c_0+b-a] + a\mathbb{Z}) = [R_{a,b,c}\tilde{u}_n, R_{a,b,c}u_n] + a\mathbb{Z}$ with $R_{a,b,c}u_n - R_{a,b,c}\tilde{u}_n = b - a$. Let u_{n+1} be the unique number in $(0, a]$ such that $R_{a,b,c}u_n - u_{n+1} \in a\mathbb{Z}$, and $\tilde{u}_{n+1} < u_{n+1}$ be the largest number such that $R_{a,b,c}\tilde{u}_n - \tilde{u}_{n+1} \in a\mathbb{Z}$. Then $(R_{a,b,c})^{n+1}([c-c_0, c-c_0+b-a] + a\mathbb{Z}) = [\tilde{u}_{n+1}, u_{n+1}] + a\mathbb{Z}$. Similarly we can show that $0 < \tilde{u}_{n+1} < u_{n+1} < a$ and $[\tilde{u}_{n+1}, u_{n+1}] \cap [c_0 + a - b, c_0] = \emptyset$, and hence $u_{n+1} = \tilde{u}_{n+1} + b - a$ and $[\tilde{u}_{n+1}, u_{n+1}] \subset (0, c_0 + a - b]$ or $[\tilde{u}_{n+1}, u_{n+1}] \subset [c_0, a)$. Now we prove that $u_{n+1} \neq c_0 + a - b$ and $\tilde{u}_{n+1} \neq c_0$. Suppose on the contrary that $u_{n+1} = c_0 + a - b$. From the inductive hypothesis, we have that for $0 \leq k \leq n$, $(R_{a,b,c})^{k+1}([0, \epsilon] + a\mathbb{Z}) = [u_k, u_k + \epsilon] + a\mathbb{Z}$ where $\epsilon > 0$ is sufficiently small. This together with (3.50), the inductive hypothesis and Proposition 3.8 implies that $([0, \epsilon] + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset$, which contradicts to (3.55) in Lemma 3.11. This proves that $u_{n+1} \neq c_0 + a - b$. Similarly we can prove that $\tilde{u}_{n+1} \neq c_0$. Hence we can proceed the inductive proof of the conclusion (5.32).

Next we prove that

$$(5.36) \quad (R_{a,b,c})^D([c-c_0, c-c_0+b-a] + a\mathbb{Z}) = [c_0 + a - b, c_0] + a\mathbb{Z}.$$

From (2.13), (5.34) and (5.35) it follows that

$$(5.37) \quad (R_{a,b,c})^D([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) = [\tilde{u}_D, u_D) + a\mathbb{Z}$$

for some $u_D \in (0, a]$ and $u_D - \tilde{u}_D = b - a$. By (5.31), $[\tilde{u}_D, u_D) \cap [c_0 + a - b, c_0) \neq \emptyset$, which implies that $u_D > c_0 + a - b$. Suppose that $c_0 + a - b < u_D < c_0$, then $(R_{a,b,c})^{D+1}([0, \epsilon) + a\mathbb{Z}) = [\tilde{u}_D, \tilde{u}_D + \epsilon) + a\mathbb{Z}$ for sufficiently small ϵ by (5.34) and (5.35). This together with (3.50), (5.34) and (5.35) and Proposition 3.8 implies that $([0, \epsilon) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset$, which contradicts to (3.55) in Lemma 3.11. Therefore $u_D \geq c_0$. Similarly, we can prove that $\tilde{u}_D \leq c_0 + a - b$. Hence (5.36) follows as $u_D - \tilde{u}_D = b - a$.

Finally we prove that $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}), 0 \leq n \leq D$, have their closures being mutually disjoint. By Proposition 3.10, we have that

$$(R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset, \quad 0 \leq n \leq D.$$

From (5.34), (5.35) and Lemma 3.11, we obtain that

$$(\tilde{u}_n - \epsilon, \tilde{u}_n) \cap \mathcal{S}_{a,b,c} \neq \emptyset \quad \text{and} \quad (u_n, u_n + \epsilon) \cap \mathcal{S}_{a,b,c} \neq \emptyset, \quad 0 \leq n \leq D$$

for any sufficiently small $\epsilon > 0$. Combining the above two observations with (5.33)–(5.36) prove mutual disjointness for the closures of holes $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}), 0 \leq n \leq D$. \square

5.2. Maximal invariant sets with rational time-frequency lattices. To prove Theorem 5.5, we need the following result about the maximality of the set $\mathcal{S}_{a,b,c}$ under the transformation $R_{a,b,c}$ in the case that $a/b \in \mathbb{Q}$, c.f. Theorem 4.1 for the maximality of the set $\mathcal{S}_{a,b,c}$ under **two** transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ in the case that $a/b \notin \mathbb{Q}$.

Lemma 5.11. *Let $a < b < c, b - a < c_0 := c - \lfloor c/b \rfloor b < a, \lfloor c/b \rfloor \geq 2, a/b = p/q$ and $c \in b\mathbb{Z}/q$ for some coprime integers p and q . Then*

- (i) $\mathcal{S}_{a,b,c}$ is the maximal set that is invariant under the transformation $R_{a,b,c}$ and that has empty intersection with the black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$ of the transformation $R_{a,b,c}$.
- (ii) $\mathbb{R} \setminus \mathcal{S}_{a,b,c}$ is the minimal set such that it is invariant under the piecewise linear transformation $R_{a,b,c}$ and that it contains the black holes of piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$.

To prove Theorem 5.5, we also need the following result about the dense property of the maximal invariant set $\mathcal{S}_{a,b,c}$ around the origin in the case that $a/b \in \mathbb{Q}$, c.f. Lemma 3.11 for the dense property of the set $\mathcal{S}_{a,b,c}$ in the case that $a/b \notin \mathbb{Q}$.

Lemma 5.12. *Let $a < b < c, b - a < c_0 < a, \lfloor c/b \rfloor \geq 2, a/b = p/q$ and $c/b \in \mathbb{Z}/q$ for some coprime integers p and q . Assume that $\mathcal{S}_{a,b,c} \neq \emptyset$. Then the following statements hold.*

- (i) *At least one of two intervals $[0, b/q)$ and $c_0 + [0, b/q)$ is contained in $\mathcal{S}_{a,b,c}$.*
- (ii) *At least one of two intervals $[-b/q, 0)$ and $c_0 + a - b + [-b/q, 0)$ is contained in $\mathcal{S}_{a,b,c}$.*
- (iii) *At least one of two intervals $[0, b/q)$ and $[-b/q, 0)$ is contained in $\mathcal{S}_{a,b,c}$.*

Now supposing that Lemmas 5.11 and 5.12 hold, we start the proof of Theorem 5.5, c.f. Examples 5.2 and 5.3 in Section 5.

Proof of Theorem 5.5. Let $\delta \in [0, c_0 + a - b]$ and $\delta' \in [c_0 - a, 0]$ be so chosen that $[\delta', \delta)$ is the maximal interval contained in $\mathbb{R} \setminus \mathcal{S}_{a,b,c}$. Then $\delta, \delta' \in b\mathbb{Z}/q$ by (5.6), and they satisfy (5.8) by the assumption $\mathcal{S}_{a,b,c} \neq \emptyset$ and Lemma 5.12, since at least one of two intervals $[0, b/q)$ and $[-b/q, 0)$ is contained in $\mathcal{S}_{a,b,c}$. Therefore the first conclusion (i) holds.

Now we divide the following three cases: (1) $\delta' = 0$ and $\delta \neq 0$; (2) $\delta = 0$ and $\delta' \neq 0$; and (3) $\delta = \delta' = 0$, to verify the conclusions (ii)–(vi).

Case 1 $\delta' = 0$ and $\delta \neq 0$

In this case,

$$(5.38) \quad [-b/q, 0) \subset \mathcal{S}_{a,b,c} \quad \text{and} \quad [0, \delta) \cap \mathcal{S}_{a,b,c} = \emptyset.$$

Then

$$(5.39) \quad [c - c_0 - b/q, c - c_0) = R_{a,b,c}[-b/q, 0) \subset R_{a,b,c}\mathcal{S}_{a,b,c} = \mathcal{S}_{a,b,c}$$

by (2.13), (5.38), and Proposition 3.8;

$$(5.40) \quad \begin{aligned} & [c - c_0 + b - a + \delta, c - c_0 + b - a + \delta + b/q) \\ &= \begin{cases} R_{a,b,c}[\delta, \delta + b/q) - a & \text{if } 0 < \delta < c_0 + a - b \\ R_{a,b,c}[c_0, c_0 + b/q) & \text{if } \delta = c_0 + a - b \end{cases} \\ &\subset R_{a,b,c}\mathcal{S}_{a,b,c} = \mathcal{S}_{a,b,c} \end{aligned}$$

by (2.13), (5.38), Proposition 3.8, the maximality of the interval $[0, \delta)$ in $\mathbb{R} \setminus \mathcal{S}_{a,b,c}$, and the first conclusion in Lemma 5.12; and

$$(5.41) \quad \begin{aligned} & [c - c_0, c - c_0 + b - a + \delta) \cap \mathcal{S}_{a,b,c} \\ &= (R_{a,b,c}[-a, \delta - a) \cup [c - c_0, c - c_0 + b - a)) \cap \mathcal{S}_{a,b,c} \\ &= R_{a,b,c}([-a, \delta - a) \cap \mathcal{S}_{a,b,c}) = \emptyset, \end{aligned}$$

where the first equality follows from (2.13), Proposition 3.8 and the assumption $[0, \delta) \subset [0, c_0 + a - b)$, and the second equality holds by Propositions 3.6 and 3.7. Thus $[c - c_0, c - c_0 + b - a + \delta)$ is a gap (i.e., an interval with empty set intersection with $\mathcal{S}_{a,b,c}$) with length $b - a + \delta$ and boundary intervals of length b/q at each side in the set $\mathcal{S}_{a,b,c}$.

Proof of the conclusion (ii). Let N_1 be the smallest nonnegative integer such that $(R_{a,b,c})^{N_1}([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}) \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) \neq \emptyset$ if it exists, and $N_1 = +\infty$ otherwise. Mimicking the argument used in the proof of Claim 5.10, we have that $N_1 < \infty$. We divide two cases to prove the conclusion (ii).

Case 1a: $N_1 = 0$.

In this case it follows from (5.39), (5.40) (5.41) that $(R_{a,b,c})^{N_1}[c - c_0, c - c_0 + b - a + \delta) = [c - c_0, c - c_0 + b - a + \delta)$ is a gap of length $b - a + \delta$ with boundary intervals of length b/q at each side in the set $\mathcal{S}_{a,b,c}$. This, together with $[c_0, c_0 + b/q) \subset \mathcal{S}_{a,b,c}$ by Lemma 5.12, and the definition of the nonnegative integer N_1 , proves that

$$(5.42) \quad (R_{a,b,c})^{N_1}[c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z} = [c_0 + a - b - \delta, c_0) + a\mathbb{Z}$$

and hence the conclusions (5.9) and (5.10) in the case that $N_1 = 0$.

Case 1b: $1 \leq N_1 < \infty$.

In this case

$$(5.43) \quad (R_{a,b,c})^n([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}) \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) = \emptyset$$

for all $0 \leq n \leq N_1 - 1$. Let us verify that

$$(5.44) \quad (R_{a,b,c})^n([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}) = [b_n + a - b - \delta, b_n) + a\mathbb{Z}$$

with

$$(5.45) \quad [b_n + a - b - \delta - b/q, b_n + b/q) \subset [0, c_0 + a - b) \cup [c_0, a),$$

$$(5.46) \quad ([b_n + a - b - \delta, b_n) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset$$

and

$$(5.47) \quad [b_n + a - b - \delta - b/q, b_n + a - b - \delta) + a\mathbb{Z}, [b_n, b_n + b/q) + a\mathbb{Z} \subset \mathcal{S}_{a,b,c}$$

for all $0 \leq n \leq N_1 - 1$. For $n = 0$, write $(R_{a,b,c})^0([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}) = [c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z} = [b_0 + a - b - \delta, b_0) + a\mathbb{Z}$ with $b_0 \in (0, a]$. Then the conclusions (5.44), (5.45), (5.46) and (5.47) for $n = 0$ follow from (3.30), (5.39), (5.40), (5.41) and (5.43). Inductively we assume that the conclusions (5.44), (5.45), (5.46) and (5.47) hold for all $0 \leq n \leq k \leq N_1 - 2$. Then for $n = k + 1$,

$$\begin{aligned} & (R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) \\ &= R_{a,b,c}[b_k + a - b - \delta, b_k) + a\mathbb{Z} \quad (\text{by (5.44) with } n = k) \\ &= [R_{a,b,c}(b_k + a - b - \delta), R_{a,b,c}(b_k + a - b - \delta) + b - a + \delta) + a\mathbb{Z} \\ & \quad (\text{by (5.45) with } n = k) \\ &=: [b_{k+1} + a - b - \delta, b_{k+1}) + a\mathbb{Z} \end{aligned}$$

for some $b_{k+1} \in (0, a]$,

$$\begin{aligned}
& ([b_{k+1} + a - b - \delta, b_{k+1}) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} \\
&= R_{a,b,c}([b_k + a - b - \delta, b_k) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} \\
&\quad (\text{by (5.43), (5.45) for } n = k, \text{ and Proposition 3.7}) \\
&= \emptyset,
\end{aligned}$$

$$\begin{aligned}
& [b_{k+1} + a - b - \delta - b/q, b_{k+1} + a - b - \delta) + a\mathbb{Z} \\
&= [R_{a,b,c}(b_k + a - b - \delta) - b/q, R_{a,b,c}(b_k + a - b - \delta)) + a\mathbb{Z} \\
&= R_{a,b,c}[b_k + a - b - \delta - b/q, b_k + a - b - \delta) + a\mathbb{Z} \\
&\quad (\text{by (5.46) with } n = k) \\
&= (R_{a,b,c})^{k+1}[c - c_0 - b/q, c - c_0) + a\mathbb{Z} \subset \mathcal{S}_{a,b,c} \quad (\text{by (5.39)})
\end{aligned}$$

and similarly

$$\begin{aligned}
& [b_{k+1}, b_{k+1} + b/q) + a\mathbb{Z} \\
&= (R_{a,b,c})^{k+1}[c - c_0 + b - a + \delta, c - c_0 + b - a + \delta + b/q) + a\mathbb{Z} \subset \mathcal{S}_{a,b,c}
\end{aligned}$$

by (5.40), (5.44), (5.45) and (5.47) for $n = k$, the definition (2.13) of the transformation $R_{a,b,c}$, and the invariance (3.30) of the set $\mathcal{S}_{a,b,c}$ under the transformation $R_{a,b,c}$. This together with (5.43) completes the inductive proof of (5.44), (5.45), (5.46), and (5.47).

Recall that $(R_{a,b,c})^{N_1}[c - c_0, c - c_0 + b - a + \delta) \cap [c_0 + a - b, c_0) + a\mathbb{Z} \neq \emptyset$ and $[c_0, c_0 + b/q) + a\mathbb{Z} \subset \mathcal{S}_{a,b,c}$ by the first conclusion in Lemma 5.12. Then applying (5.44) and (5.45) with $n = N_1 - 1$, we obtain that

$$(5.48) \quad (R_{a,b,c})^{N_1}[c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z} = [c_0 + a - b - \delta, c_0) + a\mathbb{Z},$$

because

$$\begin{aligned}
(5.49) \quad & (R_{a,b,c})^{N_1}([c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} \\
&= R_{a,b,c}([b_{N_1-1} + a - b - \delta, b_{N_1-1} + a - a) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset
\end{aligned}$$

and

$$\begin{aligned}
(5.50) \quad & [c_0 + a - b - \delta - b/q, c_0 + a - b - \delta) + a\mathbb{Z} \\
&= R_{a,b,c}([b_{N_1-1} - b + a - b/q, b_{N_1-1} - b + a) + a\mathbb{Z}) \subset \mathcal{S}_{a,b,c}.
\end{aligned}$$

Notice that

$$\begin{aligned}
(5.51) \quad & [b_n, b_n + b/q) + a\mathbb{Z} = R_{a,b,c}([b_{n-1}, b_{n-1} + b/q) + a\mathbb{Z}) \\
&= \dots = (R_{a,b,c})^n(c - c_0 + b - a + \delta) + [0, b/q) + a\mathbb{Z}.
\end{aligned}$$

The conclusion (ii), particularly the equalities in (5.9) and (5.10), in the case that $1 \leq N_1 < \infty$ then follows from (5.44), (5.48) and (5.51).

Proof of the conclusion (iii). Let N_2 be the minimal nonnegative integer such that $(R_{a,b,c})^{N_2}([c_0+a-b-\delta, c_0+a-b)+a\mathbb{Z}) \cap ([0, \delta)+a\mathbb{Z}) \neq \emptyset$ if it exists and $N_2 = +\infty$ otherwise. Mimicking the argument used to prove Claim 5.10, we have that $N_2 < \infty$. We divide three cases to verify (5.11) and (5.12).

Case 1c: $N_2 = 0$.

In this case, it follows from (5.38) and (5.50) that

$$(5.52) \quad [c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z} = [0, \delta) + a\mathbb{Z},$$

and hence (5.11) and (5.12) hold for $N_2 = 0$.

Case 1d: $N_2 = 1$.

In this case, we have that

$$(5.53) \quad R_{a,b,c}[c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z} = [\tilde{b}_1 - \delta, \tilde{b}_1) + a\mathbb{Z}$$

for some $\tilde{b}_1 \in (0, a]$ with $\tilde{b}_1 - \delta - R_{a,b,c}(c_0 + a - b - \delta) \in a\mathbb{Z}$. Recall that $[c_0 + a - b - \delta - b/q, c_0 + a - b - \delta) \subset \mathcal{S}_{a,b,c}$ and $[c_0, c_0 + b/q) \subset \mathcal{S}_{a,b,c}$ by (5.38), (5.50) and Lemma 5.12, we then obtain from (5.53) and Proposition 3.8 that

$$(5.54) \quad [\tilde{b}_1 - \delta - b/q, \tilde{b}_1 - \delta) + a\mathbb{Z} \subset \mathcal{S}_{a,b,c} \text{ and } [\tilde{b}_1, \tilde{b}_1 + b/q) + a\mathbb{Z} \subset \mathcal{S}_{a,b,c}.$$

Therefore $[\tilde{b}_1 - \delta, \tilde{b}_1)$ is a gap of length δ with boundary intervals of length b/q at each side in the set $\mathcal{S}_{a,b,c}$. Thus

$$[\tilde{b}_1 - \delta, \tilde{b}_1) \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) = \emptyset$$

as the gap containing $[c_0 + a - b, c_0)$ is $(R_{a,b,c})^{N_1}[c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z}$ which has length $b - a + \delta$ and boundary intervals of length b/q at each side in $\mathcal{S}_{a,b,c}$. By the definition of the nonnegative integer N_2 , $([\tilde{b}_1 - \delta, \tilde{b}_1) + a\mathbb{Z}) \cap ([0, \delta) + a\mathbb{Z}) \neq \emptyset$. This together with (5.54) and $[-b/q, 0) \in \mathcal{S}_{a,b,c}$ implies that $\tilde{b}_1 = \delta$ and

$$(5.55) \quad R_{a,b,c}[c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z} = [0, \delta) + a\mathbb{Z}.$$

The conclusion (5.11) and (5.12) for $N_2 = 1$ follow from (5.53) and (5.55).

Case 1e: $2 \leq N_2 < +\infty$.

In this case, following the arguments for the first conclusion of this theorem and the arguments in the case that $N_2 = 1$, we may inductively show that

$$(5.56) \quad (R_{a,b,c})^m([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z}) = [\tilde{b}_m - \delta, \tilde{b}_m) + a\mathbb{Z}, 1 \leq m \leq N_2 - 1,$$

for some $\tilde{b}_m \in (0, a]$ with $[\tilde{b}_m - \delta - b/q, \tilde{b}_m + b/q) \subset [0, c_0 + a - b) \cup [c_0, a)$, $[\tilde{b}_m - \delta - b/q, \tilde{b}_m - \delta) \subset \mathcal{S}_{a,b,c}$ and $[\tilde{b}_m, \tilde{b}_m + b/q) \subset \mathcal{S}_{a,b,c}$. Applying

(5.56) with $m = N_2 - 1$, and recalling the definition of the integer N_2 we obtain that

$$(5.57) \quad (R_{a,b,c})^{N_2}([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z}) = [0, \delta) + a\mathbb{Z}$$

and

$$(5.58) \quad [-b/q, 0), [\delta, \delta + b/q) \in \mathcal{S}_{a,b,c}.$$

Therefore the conclusions (5.11) and (5.12) for $2 \leq N_2 < \infty$ follow from (5.56), (5.57) and (5.58).

Proof of the conclusion (iv). First we prove the mutually disjoint property for $(R_{a,b,c})^n([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}), 0 \leq n \leq N_1$, i.e.,

$$(5.59) \quad (R_{a,b,c})^{n_1}([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}) \cap (R_{a,b,c})^{n_2}([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}) = \emptyset$$

for all $0 \leq n_1 \neq n_2 \leq N_1$. By (5.44)–(5.50) and Proposition 3.10, $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a + \delta), 0 \leq n \leq N_1$, are gaps with length $b - a + \delta$ and boundary intervals of length b/q at each side contained in the set $\mathcal{S}_{a,b,c}$. Then for any $0 \leq n_1 \neq n_2 \leq N_1$, either $(R_{a,b,c})^{n_1}([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}) = (R_{a,b,c})^{n_2}([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z})$ or $(R_{a,b,c})^{n_1}([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}) \cap (R_{a,b,c})^{n_2}([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}) = \emptyset$. Suppose that $(R_{a,b,c})^{n_1}([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z}) = (R_{a,b,c})^{n_2}([c - c_0, c + b - c_0 - a + \delta) + a\mathbb{Z})$ for some $0 \leq n_1 < n_2 \leq N_1$. Then $n_2 \leq N_1 - 1$ by (5.43) and (5.48). Thus by (5.43) and the one-to-one correspondence of the transformation $R_{a,b,c}$ on the complement of $[c_0 + a - b, c_0) + a\mathbb{Z}$, we have that $(R_{a,b,c})^{n_2 - n_1}([c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z}) = [c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z}$, which contradicts to the range property (3.49) as $((R_{a,b,c})^{n_2 - n_1}([c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z}) \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) = \emptyset$ by (5.43). This proves (5.59).

Next we verify that $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z}), 0 \leq m \leq N_2$, are mutually disjoint when $1 \leq N_2 < \infty$. Recall that $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z}), 1 \leq m \leq N_2$, are gaps of length δ with boundary intervals of length b/q on each side contained in the set $\mathcal{S}_{a,b,c}$ by (5.55), (5.56) and (5.57). Therefore any two of gaps $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z}), 1 \leq m \leq N_2$, are either identical or has empty-set intersection. If there exist $1 \leq m_1 < m_2 \leq N_2$ such that gaps $(R_{a,b,c})^{m_1}([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z})$ and $(R_{a,b,c})^{m_2}([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z})$ are identical, then the front parts of those gaps of length $\min(\delta, b - a)$ should be identical too, i.e., $(R_{a,b,c})^{m_1}([c_0 + a - b - \delta, c_0 + a - b - \delta + \min(\delta, b - a)) + a\mathbb{Z}) = (R_{a,b,c})^{m_2}([c_0 + a - b - \delta, c_0 + a - b - \delta + \min(\delta, b - a)) + a\mathbb{Z})$. Recall from (5.42) and (5.48) that $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 + a - b - \delta + \min(\delta, b - a) + a\mathbb{Z}) = (R_{a,b,c})^{m + N_1}([c - c_0, c - c_0 + \min(\delta, b - a)) + a\mathbb{Z}), 0 \leq m \leq N_2$ and $(R_{a,b,c})^n([c - c_0, c - c_0 + \min(\delta, b - a)) + a\mathbb{Z}) \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) = \emptyset$

for all $0 \leq n \leq N_1 + N_2$. Therefore by the one-to-one correspondence of the transformation $R_{a,b,c}$ on the complement of its black hole in Proposition 3.7, $(R_{a,b,c})^{m_2-m_1}[c - c_0, c - c_0 + \min(\delta, b - a)) + a\mathbb{Z} = [c - c_0, c - c_0 + \min(\delta, b - a)) + a\mathbb{Z}$, which contradicts to the range property (3.49) of the transformation $R_{a,b,c}$. This proves that $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z})$, $1 \leq m \leq N_2$, are mutually disjoint.

From the above argument, we see that $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z})$, $0 \leq n \leq N_1$, are mutually disjoint, and that $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z})$, $1 \leq m \leq N_2$, are mutually disjoint. Then verification of the mutually disjoint property in the conclusion (iv) reduces to showing that any gap of the form $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z})$, $0 \leq n \leq N_1$, has empty intersection with any gap of the form $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z})$, $1 \leq m \leq N_2$. This is true because $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a + \delta), 0 \leq n \leq N_1 - 1$, are gaps of length $b - a + \delta$ with boundary intervals of length b/q at each side contained in the set $\mathcal{S}_{a,b,c}$ by (5.44)–(5.50) and $(R_{a,b,c})^m([c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z})$, $1 \leq m \leq N_2$, are gaps of length δ with boundary intervals of length b/q on each side contained in the set $\mathcal{S}_{a,b,c}$ by (5.55), (5.56) and (5.57).

Proof of the conclusion (v). Write $\delta = l(b - a) + \tilde{\delta}$ for some $0 \leq l \in \mathbb{Z}$ and $\tilde{\delta} \in (0, b - a]$. From (5.9)–(5.12), we obtain that

$$\begin{aligned} & (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \\ = & \begin{cases} b_{n-\tilde{l}(N_1+N_2+1)} + \tilde{l}(b - a) - \delta + [a - b, 0) + a\mathbb{Z} \\ \quad \text{if } 0 \leq n - \tilde{l}(N_1 + N_2 + 1) \leq N_1 \text{ for some } 0 \leq \tilde{l} \leq l, \\ \tilde{b}_{n-\tilde{l}(N_1+N_2+1)-N_1} + \tilde{l}(b - a) - \delta + [a - b, 0) + a\mathbb{Z} \\ \quad \text{if } 1 \leq n - \tilde{l}(N_1 + N_2 + 1) - N_1 \leq N_2 \text{ for some } 0 \leq \tilde{l} \leq l - 1, \\ ((\tilde{b}_{n-l(N_1+N_2+1)-N_1} + [-\tilde{\delta}, 0)) \cup [c_0 + a - b, c_0 - \tilde{\delta})) + a\mathbb{Z} \\ \quad \text{if } 1 \leq n - l(N_1 + N_2 + 1) - N_1 \leq N_2, \\ ((b_{n-(l+1)(N_1+N_2+1)} + [-\tilde{\delta}, 0)) \cup [c_0 + a - b, c_0 - \tilde{\delta})) + a\mathbb{Z} \\ \quad \text{if } 0 \leq n - (l + 1)(N_1 + N_2 + 1) \leq N_1, \end{cases} \end{aligned}$$

where $b_n = (R_{a,b,c})^n(c - c_0 + b - a + \delta)$, $0 \leq n \leq N_1$, and $\tilde{b}_m = (R_{a,b,c})^m(c_0)$, $1 \leq m \leq N_2$, c.f. Example 5.2 in Section 5. Therefore

$$\begin{aligned} & (\cup_{n=0}^{N_1} ((R_{a,b,c})^n[c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z}) \\ & \quad \cup (\cup_{m=1}^{N_2} (R_{a,b,c})^m[c_0 + a - b - \delta, c_0 + a - b) + a\mathbb{Z})) \\ & = \cup_{n=0}^{N_1+N_2} (R_{a,b,c})^n([c - c_0, c - c_0 + b - a + \delta) + a\mathbb{Z}) \\ (5.60) \quad & = \cup_{n=0}^{(l+1)(N_1+N_2+1)+N_1} (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}), \end{aligned}$$

and

(5.61)

$$(R_{a,b,c})^{(l+1)(N_1+N_2+1)+N_1}[c-c_0, c-c_0+b-a)+a\mathbb{Z}] = [c_0+a-b, c_0)+a\mathbb{Z}.$$

Hence the union of the gaps $((R_{a,b,c})^n[c-c_0, c-c_0+b-a+\delta)+a\mathbb{Z})$, $0 \leq n \leq N_1$, and $(R_{a,b,c})^m[c_0+a-b-\delta, c_0+a-b)+a\mathbb{Z})$, $1 \leq m \leq N_2$, is invariant under the transformation $R_{a,b,c}$ and contains the black holes of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$. It also indicates that any points not in that union will not be in that union under the transformation $R_{a,b,c}$. This together with Lemma 5.11 proves (5.13) and hence the conclusion (v).

Proof of the conclusion (vi). From the arguments to prove the conclusions (ii) and (iii), the boundary intervals of the mutually disjoint gaps $((R_{a,b,c})^n[c-c_0, c-c_0+b-a+\delta)+a\mathbb{Z})$, $0 \leq n \leq N_1$ and $(R_{a,b,c})^m[c_0+a-b-\delta, c_0+a-b)+a\mathbb{Z})$, $1 \leq m \leq N_2$, of length b/q at each side are contained in the set $\mathcal{S}_{a,b,c}$. Therefore the set $\mathcal{S}_{a,b,c}$ is the union of intervals $[b_n, b_n+h_n)+a\mathbb{Z}$, $0 \leq n \leq N_1$ and $[\tilde{b}_m, \tilde{b}_m+\tilde{h}_m)+a\mathbb{Z}$, where $0 < h_n \in b\mathbb{Z}/q$, $0 \leq n \leq N_1$, and $0 < \tilde{h}_m \in a\mathbb{Z}/q$, $1 \leq m \leq N_2$, are chosen so that $[b_n+h_n, b_n+h_n+b/q)+a\mathbb{Z}$, $0 \leq n \leq N_1$ and $[\tilde{b}_m+\tilde{h}_m, \tilde{b}_m+\tilde{h}_m+b/q)+a\mathbb{Z}$, $1 \leq m \leq N_2$, are contained in gaps. As $[0, \delta)+a\mathbb{Z}$ and $[c_0+a-b, c_0)+a\mathbb{Z}$ are contained in the union of the mutually disjoint gaps, each of the intervals $[b_n, b_n+h_n)+a\mathbb{Z}$, $0 \leq n \leq N_1$, and $[\tilde{b}_m, \tilde{b}_m+\tilde{h}_m)+a\mathbb{Z}$, $1 \leq m \leq N_2$, is contained either in $[0, c_0+a-b)+a\mathbb{Z}$ or in $[c_0, a)+a\mathbb{Z}$, and its boundary interval of length b/q at each side is not contained in the set $\mathcal{S}_{a,b,c}$. Recall that $b_n - (R_{a,b,c})^n(c-c_0+b-a+\delta) \in a\mathbb{Z}$, $0 \leq n \leq N_1$, and $\tilde{b}_m - (R_{a,b,c})^{m+N_1}(c-c_0+b-a+\delta) \in a\mathbb{Z}$, $1 \leq m \leq N_2$, from the second and third conclusions of this theorem. Hence the interval $[b_n, b_n+h_n)+a\mathbb{Z} = (R_{a,b,c})^n[b_0, b_0+h_0)+a\mathbb{Z}$ and $[\tilde{b}_m, \tilde{b}_m+\tilde{h}_m)+a\mathbb{Z} = (R_{a,b,c})^{m+N_1}[b_0, b_0+h_0)+a\mathbb{Z}$. This together with the measure-preserving property in Proposition 3.8 implies that the length of intervals contained in the set $\mathcal{S}_{a,b,c}$ are the same, i.e.,

$$(5.62) \quad h_n = \tilde{h}_m = h \text{ for all } 0 \leq n \leq N_1 \text{ and } 1 \leq m \leq N_2$$

where $0 < h \in a\mathbb{Z}/q$. Hence (5.14) holds. Finally we verify (5.15). Note that the measure of the gaps contained in $[0, a)$ is equal to $(N_1+1)(b-a+\delta) + N_2\delta$, while the measure of the intervals contained in $\mathcal{S}_{a,b,c} \cap [0, a)$ is $(N_1+N_2+1)h$. Therefore

$$(5.63) \quad (N_1+N_2+1)h + (N_1+1)(b-a+\delta) + N_2\delta = a.$$

Thus (5.15) follows. This completes the proof of the conclusion (ii)–(vi) for the case that $\delta' = 0$ and $\delta \neq 0$.

Case 2 $\delta = 0$ and $\delta' \neq 0$.

We follow the argument used for Case 1 to prove the desired conclusion. We omit the details here.

Case 3 $\delta = \delta' = 0$.

In this case, we have that

$$(5.64) \quad [-b/q, b/q) \subset \mathcal{S}_{a,b,c}.$$

Then

$[c - c_0 - b/q, c - c_0) \subset \mathcal{S}_{a,b,c}$ and $[c + b - c_0 - a, c + b - c_0 - a + b/q) \subset \mathcal{S}_{a,b,c}$ as $[c - c_0 - b/q, c - c_0) = R_{a,b,c}[-b/q, 0)$ and $[c + b - c_0 - a, c + b - c_0 - a + b/q) = R_{a,b,c}[0, b/q)$ and the set $\mathcal{S}_{a,b,c}$ is invariant under the transformation $R_{a,b,c}$ by Proposition 3.8.

Proof of the conclusion (ii). Let N_1 be the smallest nonnegative integer such that $(R_{a,b,c})^{N_1}([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) \neq \emptyset$ if it exists, and $N_1 = +\infty$ otherwise. Following the argument in Case 1, we can show that $N_1 < +\infty$, and

$$(5.65) \quad (R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z}), 0 \leq n \leq N_1, \text{ are mutually disjoint,}$$

$$(5.66) \quad (R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) = [b_n - b + a, b_n) + a\mathbb{Z}$$

for some $b_n \in (0, a)$ with $b_n - (R_{a,b,c})^n(c + b - c_0 - a) \in a\mathbb{Z}$, $[b_n - b + a - b/q, b_n + b/q) \subset [0, c_0 + a - b) \cup [c_0, a)$,

$$(5.67) \quad [b_n + a - b - b/q, b_n + a - b) + a\mathbb{Z}, [b_n, b_n + b/q) + a\mathbb{Z} \subset \mathcal{S}_{a,b,c}$$

for $0 \leq n \leq N_1$, and

$$(5.68) \quad [b_{N_1} + a - b, b_{N_1}) = [c_0 + a - b, c_0).$$

Therefore the conclusion (ii) follows.

Proof of the conclusion (iii)'. Let N_2 be the smallest positive integer such that $(R_{a,b,c})^{N_2}(c_0) \in a\mathbb{Z}$ if it exists and $N_2 = +\infty$ otherwise. To prove (iii)', it suffices to prove that

$$(5.69) \quad N_2 < +\infty.$$

To prove (5.69), we need the following mutually disjoint property when $N_2 < \infty$:

$$(5.70) \quad (R_{a,b,c})^m(c_0) + [-b/(2q), b/(2q)) + a\mathbb{Z}, 0 \leq m \leq N_2, \text{ are mutually disjoint.}$$

Proof. Suppose, on the contrary, that the mutual disjoint property in (5.70) does not hold. Then there exist $1 \leq m_1 < m_2 \leq N_2$ such that

$$(5.71) \quad (R_{a,b,c})^{m_1}(c_0) + a\mathbb{Z} = (R_{a,b,c})^{m_2}(c_0) + a\mathbb{Z}.$$

This implies that $m_2 < N_2$ by the definition of the integer N_2 . Recall that $c_0 \in \mathcal{S}_{a,b,c}$ by applying (5.67) and (5.68) with $n = N_1$. Then $(R_{a,b,c})^{m_2}(c_0) \in \mathcal{S}_{a,b,c}$ by Proposition 3.8. This, together with the one-to-one correspondence for the transformation $R_{a,b,c}$ onto $\mathcal{S}_{a,b,c}$ given in Proposition 3.8, leads to

$$(5.72) \quad c_0 + a\mathbb{Z} = (R_{a,b,c})^{m_2-m_1}(c_0) + a\mathbb{Z}$$

by (5.71). Recall that $c_0 \in (R_{a,b,c})^{N_1+1}(0) + a\mathbb{Z}$. Then applying the one-to-one correspondence of the transformation $R_{a,b,c}$ on the invariant set $\mathcal{S}_{a,b,c}$ again, we obtain from (5.72) that $(R_{a,b,c})^{m_2-m_1}(0) \in a\mathbb{Z}$. This is a contradiction as $m_2 - m_1 \leq N_1 - 1$ and $N_1 + N_2 + 1$ is the smallest positive integer n such that $(R_{a,b,c})^n(0) \in a\mathbb{Z}$. \square

Now we prove that $N_2 \leq p - 1$. Suppose on the contrary that $N_2 \geq p$. Following the argument in the proof of (5.70), we obtain that $(R_{a,b,c})^m(c_0) + [-b/(2q), b/(2q)) + a\mathbb{Z}, 0 \leq m \leq p$ are mutually disjoint. This is a contradiction as there are at most p elements in the set $[0, a) \cap b\mathbb{Z}/q$, and hence proves the conclusion (iii)'.

Proof of the conclusion (iv)'. By (5.65) and (5.70), verification of the mutually joint property in the conclusion (iv)' reduces to showing that $(R_{a,b,c})^m(c_0) \notin \cup_{n=0}^{N_1} (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z})$ for all $0 \leq m \leq N_2$, which is true as $(R_{a,b,c})^m(c_0) \in \mathcal{S}_{a,b,c}$ for all $m \geq 0$ and $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) \cap \mathcal{S}_{a,b,c} = \emptyset$ for all $0 \leq n \leq N_1$.

Proof of the conclusion (v). We can follow the argument in the first case $\delta > 0$ and $\delta' = 0$, as $\cup_{n=0}^{N_1} (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z})$ is invariant under the transformation $R_{a,b,c}$ and contains the black holes of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$. We omit the details here.

Proof of the conclusion (vi). By (5.66)–(5.68), gaps of length $b - a$ are $(R_{a,b,c})^n(c + b - c_0 - a) + [a - b, 0) + a\mathbb{Z}, 0 \leq n \leq N_1$, while gaps of length zero are located at $(R_{a,b,c})^m(c_0) + a\mathbb{Z} = (R_{a,b,c})^{m+N_1}(c + b - c_0 - a) + a\mathbb{Z}, 1 \leq m \leq N_2$. By the conclusions (iii)' and (v), we can divide the set $\mathcal{S}_{a,b,c}$ as the union of disjoint union of intervals who are left-closed and right-open such that each interval has its left endpoint being the same of the right endpoint of a gap, each interval has its right endpoint being the same as the left endpoint of a gap, and each interval

has its interior not containing the location of any gaps of length zero; that is

$$(5.73) \quad \mathcal{S}_{a,b,c} = \cup_{n=0}^{N_1+N_2} [b_n, b_n + h_n) + a\mathbb{Z},$$

where

$$(5.74) \quad b_n \in (R_{a,b,c})^n(c + b - c_0 - a) + a\mathbb{Z}$$

and

$$(5.75) \quad b_n + h_n \in \{b_{n_1} + a - b + a\mathbb{Z}\}_{n_1=0}^{N_1} \cup \{(R_{a,b,c})^{m_1}(c_0) + a\mathbb{Z}\}_{m_1=1}^{N_2}$$

for all $0 \leq n \leq N_1 + N_2$, and

$$(5.76) \quad (R_{a,b,c})^m(c_0) \notin \cup_{n=0}^{N_1+N_2} (b_n, b_n + h_n) + a\mathbb{Z}$$

for all $1 \leq m \leq N_2$. Therefore the proof of (5.14) reduces to establishing

$$(5.77) \quad h_n = h_0, \quad 0 \leq n \leq N_1 + N_2.$$

By the measure-preserving property in Proposition 3.8 it suffices to prove that

$$(5.78) \quad [b_{n+1}, b_{n+1} + h_{n+1}) + a\mathbb{Z} \subset R_{a,b,c}([b_n, b_n + h_n) + a\mathbb{Z}), \quad 0 \leq n \leq N_1 + N_2 - 1$$

and

$$(5.79) \quad [b_n, b_n + h_n) + a\mathbb{Z} \subset \tilde{R}_{a,b,c}([b_{n+1}, b_{n+1} + h_{n+1}) + a\mathbb{Z}), \quad 0 \leq n \leq N_1 + N_2 - 1.$$

By (5.74) and the measure-preserving property in Proposition 3.8, we have that $R_{a,b,c}([b_n, b_n + h_n) + a\mathbb{Z}) = [b_{n+1}, b_{n+1} + h_n) + a\mathbb{Z}$. Observe that for each $0 \leq n \leq N_1 + N_2$, $b_{n+1} + h_n$ is the left endpoint of a gap because

$$b_{n+1} + h_n = \begin{cases} R_{a,b,c}(b_n + h_n) & \text{if } b_n + h_n \notin \{0, c_0 + a - b\} + a\mathbb{Z} \\ R_{a,b,c}(c_0) & \text{if } b_n + h_n \in c_0 + a - b + a\mathbb{Z}, \end{cases}$$

and $b_n + h_n \notin b_{N_1+N_2} + h_{N_1+N_2} + a\mathbb{Z} = a\mathbb{Z}$ for all $0 \leq n \leq N_1 + N_2 - 1$ by the conclusion (iv)'. This together with (5.76) and the fact that $R_{a,b,c}([b_n, b_n + h_n) + a\mathbb{Z}) \subset \mathcal{S}_{a,b,c}$ proves (5.78).

Similarly, by (5.74) and the measure-preserving property in Proposition 3.8, we have that $0 \leq n \leq N_1 + N_2 - 1$, $\tilde{R}_{a,b,c}([b_{n+1}, b_{n+1} + h_{n+1}) + a\mathbb{Z}) = [b_n, b_n + h_{n+1}) + a\mathbb{Z}$ and $b_n + h_{n+1}$ is the left endpoint of a gap because

$$b_n + h_{n+1} = \begin{cases} \tilde{R}_{a,b,c}(b_{n+1} + h_{n+1}) & \text{if } b_{n+1} + h_{n+1} \notin \{c, c - c_0\} + a\mathbb{Z} \\ \tilde{b}_{N_1} & \text{if } b_{n+1} + h_{n+1} \in c + a\mathbb{Z}, \end{cases}$$

and $b_{n+1} + h_{n+1} \notin c - c_0 + a\mathbb{Z} = b_0 - b + a + a\mathbb{Z}$ for all $0 \leq n \leq N_1 + N_2 - 1$ by the conclusion (iv)'. Thus (5.79) follows. This proves (5.77) and hence (5.14).

The equation (5.15) holds by (5.14) and the conclusion (iv) in this theorem. \square

Now it remains to prove Lemmas 5.11 and 5.12.

Proof of Lemma 5.11. (i) Clearly it suffices to prove that

$$(5.80) \quad \mathcal{S}_{a,b,c} = (R_{a,b,c})^D \mathbb{R} \setminus ([c_0 + a - b, c_0) + a\mathbb{Z})$$

for some nonnegative integer D . For $n \geq 0$, write

$$(5.81) \quad (R_{a,b,c})^n \mathbb{R} \setminus ([c_0 + a - b, c_0) + a\mathbb{Z}) = A_n + [0, b/q) + a\mathbb{Z}$$

where $A_n \subset \{0, b/q, \dots, (p-1)b/q\} \setminus [c_0 + a - b, c_0)$. The existence of such a finite set A_n follows from the assumption on the triple (a, b, c) and the definition (2.13) of the transformation $R_{a,b,c}$. Clearly,

$$(5.82) \quad A_{n+1} \subset A_n, \quad n \geq 0.$$

Notice that there are at most $2p - q$ elements in the set $\{0, b/q, \dots, (p-1)b/q\} \setminus [c_0 + a - b, c_0)$. This together with (5.82) implies that

$$(5.83) \quad A_{D+1} = A_D$$

for some nonnegative integer $D \leq 2p - q$. Without loss of generality, we assume that D is the smallest nonnegative integer such that (5.83) holds. By Propositions 3.8 and 3.6, we have that $\mathcal{S}_{a,b,c} = (R_{a,b,c})^D \mathcal{S}_{a,b,c} \subset (R_{a,b,c})^D \mathbb{R}$ and $\mathcal{S}_{a,b,c} \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) = \emptyset$. This leads to the following inclusion

$$(5.84) \quad \mathcal{S}_{a,b,c} \subset A_D + [0, b/q) + a\mathbb{Z}.$$

Clearly the conclusion (5.80) follows in the case that $A_D = \emptyset$. Now we consider the case that $A_D \neq \emptyset$ and prove

$$(5.85) \quad A_D + a\mathbb{Z} \subset \mathcal{S}_{a,b,c}.$$

For $n \geq 0$, it follows from (2.13) and (5.81) that

$$\begin{aligned} & A_{n+1} + [0, b/q) + a\mathbb{Z} \\ &= R_{a,b,c}((R_{a,b,c})^n \mathbb{R} \setminus ([c_0 + a - b, c_0) + a\mathbb{Z})) \setminus ([c_0 + a - b, c_0) + a\mathbb{Z}) \\ &= R_{a,b,c}(A_n + [0, b/q) + a\mathbb{Z}) \setminus ([c_0 + a - b, c_0) + a\mathbb{Z}) \\ &= R_{a,b,c}(A_n + a\mathbb{Z}) \setminus ([c_0 + a - b, c_0) + a\mathbb{Z}) + [0, b/q). \end{aligned}$$

Therefore

$$(5.86) \quad A_{n+1} + a\mathbb{Z} = R_{a,b,c}(A_n + a\mathbb{Z}) \setminus ([c_0 + a - b, c_0) + a\mathbb{Z}), \quad n \geq 0.$$

Applying (5.86) with $n = D$ and using (5.83), we obtain that

$$A_D + a\mathbb{Z} = A_{D+1} + a\mathbb{Z} = R_{a,b,c}(A_D + a\mathbb{Z}) \setminus ([c_0 + a - b, c_0) + a\mathbb{Z}).$$

Thus

$$(5.87) \quad R_{a,b,c}(A_D) + a\mathbb{Z} = A_D + a\mathbb{Z}$$

as the cardinality of the sets A_D and $R_{a,b,c}(A_D + a\mathbb{Z}) \cap [0, a)$ are the same. This together with (3.49) and (5.86) implies that

$$(5.88) \quad (A_D + a\mathbb{Z}) \cap ([c - c_0, c + b - c_0 + a) + a\mathbb{Z}) = \emptyset.$$

Hence

$$(5.89) \quad \tilde{R}_{a,b,c}(A_D) + a\mathbb{Z} = A_D + a\mathbb{Z}$$

by (5.81), (5.87), (5.88) and Proposition 3.8.

Take $t \in A_D$. Then $(R_{a,b,c})^n(t)$ and $(\tilde{R}_{a,b,c})^n(t)$, $n \geq 0$, belong to the set $A_D + a\mathbb{Z}$ by (5.87) and (5.89), and hence they do not belong to the black holes of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$. Therefore $t \in \mathcal{S}_{a,b,c}$ by Proposition 4.3 and hence (5.85) is established.

Notice that

$$(5.90) \quad A_D + [0, b/q) + a\mathbb{Z} \subset \mathcal{S}_{a,b,c}$$

by (5.5) and (5.85). Then in the case that $A_D \neq \emptyset$, the conclusion (5.80) follows from (5.81), (5.84) and (5.91).

(ii) The desired minimality follows from

$$(5.91) \quad \mathbb{R} \setminus \mathcal{S}_{a,b,c} = \cup_{n=0}^{\infty} (R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z}).$$

By Proposition 3.10,

$$\cup_{n=0}^{\infty} (R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) \subset \mathbb{R} \setminus \mathcal{S}_{a,b,c}.$$

Then it suffices to prove

$$(5.92) \quad \mathbb{R} \setminus \mathcal{S}_{a,b,c} \subset \cup_{n=0}^L (R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z})$$

with

$$(5.93) \quad (R_{a,b,c})^L([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) = [c_0 + a - b, c_0) + a\mathbb{Z}$$

for some nonnegative integer $L \geq 0$, c.f. (5.2) and (5.4) in Theorem 5.4. First we prove that

$$(5.94) \quad (R_{a,b,c})^L([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) \subset [c_0 + a - b, c_0) + a\mathbb{Z}$$

for some nonnegative integer L . By (5.6) and finite cardinality of the set $b\mathbb{Z}/q \cap [0, a)$, we only need to verify that for any $t \in [c - c_0, c + b - c_0 - a) + a\mathbb{Z}) \cap b\mathbb{Z}/q$ there exists L_0 such that $(R_{a,b,c})^{L_0}(t) \in [c_0 + a - b, c_0) + a\mathbb{Z}$. Suppose on the contrary that $(R_{a,b,c})^n(t) \notin [c_0 + a - b, c_0) + a\mathbb{Z}$ for all $n \geq 0$. Then $(R_{a,b,c})^n(t) \notin [c - c_0, c + b - c_0 - a) + a\mathbb{Z}$, $n \geq 1$,

by Proposition 3.7, which together with finite cardinality of the set $b\mathbb{Z}/q \cap [0, a)$ implies the existences of positive integers $1 \leq L_1 < L_2$ such that $(R_{a,b,c})^{L_1}(t) - (R_{a,b,c})^{L_2}(t) \in a\mathbb{Z}$. Set $t_0 = (R_{a,b,c})^{L_1}(t)$. Then $(R_{a,b,c})^n(t_0)$ and $(\tilde{R}_{a,b,c})^n(t_0)$, $n \geq 1$, do not belong to black holes of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ as $(R_{a,b,c})^n(t_0) - (R_{a,b,c})^{m+L_1}(t) \in a\mathbb{Z}$ and $(\tilde{R}_{a,b,c})^{\tilde{n}}(t_0) - (R_{a,b,c})^{L_2-\tilde{m}}(t) \in a\mathbb{Z}$ where $m = n - \lfloor n/(L_2 - L_1) \rfloor (L_2 - L_1)$, and $\tilde{m} = \tilde{n} - \lfloor \tilde{n}/(L_2 - L_1) \rfloor (L_2 - L_1)$. Thus $t_0 \in \mathcal{S}_{a,b,c}$ by Proposition 3.52, which contradicts to (5.92).

Next we prove (5.93). By (5.94), it suffices to prove that

$$(R_{a,b,c})^L(t_1) - (R_{a,b,c})^L(t_2) \notin a\mathbb{Z}$$

for any distinct $t_1, t_2 \in ([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) \cap b\mathbb{Z}/q$. Let k_1, k_2 be minimal nonnegative integers such that

$$(R_{a,b,c})^{k_1}(t_1) = (R_{a,b,c})^L(t_1) \quad \text{and} \quad (R_{a,b,c})^{k_2}(t_2) = (R_{a,b,c})^L(t_2).$$

Without loss of generality, we assume that $k_1 \leq k_2$. By one-to-one correspondence of the transformation $R_{a,b,c}$ given in Proposition 3.7 and the selection of integers k_1 and k_2 ,

$$t_1 = (\tilde{R}_{a,b,c})^{k_1}((R_{a,b,c})^{k_1}(t_1)) \in (R_{a,b,c})^{k_2-k_1}(t_2) + a\mathbb{Z},$$

which is a contradiction by the assumption $t_1 \notin t_2 + a\mathbb{Z}$ (if $k_2 = k_1$) and the range property of the transformation $R_{a,b,c}$ in Proposition 3.7 (if $k_2 > k_1$). Hence (5.93) is proved.

By (5.93), $\mathbb{R} \setminus (\cup_{n=0}^L (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}))$ has empty intersection with $[c_0 + a - b, c_0) + a\mathbb{Z}$, the black hole of the transformation $R_{a,b,c}$. Then by the first conclusion of this lemma, the proof of (5.92) reduces to the invariance of the set $\mathbb{R} \setminus (\cup_{n=0}^L (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z}))$ under the transformation $R_{a,b,c}$. Suppose, on the contrary, that there exists $t \notin \cup_{n=0}^L (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z})$ such that $R_{a,b,c}(t) \in \cup_{n=0}^L (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z})$. Then $t \notin [c_0 + a - b, c_0) + a\mathbb{Z}$ by (5.93), and $R_{a,b,c}(t) = (R_{a,b,c})^n(s)$ for some $0 \leq n \leq L$ and $s \in [c - c_0, c - c_0 + b - a) + a\mathbb{Z}$. It follows that $n \geq 1$ from the range property of the transformation $R_{a,b,c}$ in Proposition 3.7. If we further select the integer n to be the smallest positive integer such that $R_{a,b,c}(t) = (R_{a,b,c})^n(s)$ for some $s \in [c - c_0, c - c_0 + b - a) + a\mathbb{Z}$. Then $t = (R_{a,b,c})^{n-1}(s)$ by the one-to-one correspondence of the transformation $R_{a,b,c}$ given in Proposition 3.7, which contradicts to the assumption that $t \notin \cup_{n=0}^L (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z})$. \square

We finish this subsection by the proof of Lemma 5.12.

Proof of Lemma 5.12. (i) Suppose on the contrary that both $[0, b/q)$ and $[c_0, c_0 + b/q)$ are not contained in $\mathcal{S}_{a,b,c}$. Then $0, c_0 \notin \mathcal{S}_{a,b,c}$ by (5.6),

which together with (5.6) and Proposition 3.6 implies that

$$(5.95) \quad \mathcal{S}_{a,b,c} \subset ([b/q, c_0 + a - b) \cup [c_0 + b/q, a)) + a\mathbb{Z}.$$

Thus

$$(5.96) \quad R_{a,b,c}(\mathcal{S}_{a,b,c} - b/q) \subset \mathcal{S}_{a,b,c} - b/q$$

as $R_{a,b,c}(t - b/q) = R_{a,b,c}(t) - b/q$ for all $t \in \mathcal{S}_{a,b,c} \subset ([b/q, c_0 + a - b) \cup [c_0 + b/q, a)) + a\mathbb{Z}$. Thus both $\mathcal{S}_{a,b,c}$ and $\mathcal{S}_{a,b,c} - b/q$ are invariant under the transformation $R_{a,b,c}$ and have empty intersection with the black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$ of the transformation $R_{a,b,c}$ by (5.96) and Proposition 3.8. Hence by the maximality of the set $\mathcal{S}_{a,b,c}$ in Lemma 5.11, $\mathcal{S}_{a,b,c} - b/q \subset \mathcal{S}_{a,b,c}$, which contradicts to (5.95) because $t_0 - b/q \in \mathcal{S}_{a,b,c} - b/q$ but $t_0 - b/q \notin \mathcal{S}_{a,b,c}$ by (5.6) and (5.95) where t_0 is the smallest positive number in $\mathcal{S}_{a,b,c} \cap [0, a)$.

(ii) Suppose on the contrary that both $[-b/q, 0)$ and $[c_0 + a - b - b/q, c_0 + a - b)$ are not contained in $\mathcal{S}_{a,b,c}$. Then $a - b/q, c_0 + a - b - b/q \notin \mathcal{S}_{a,b,c}$ by (5.6). Following the above argument in the proof of the first conclusion, $R_{a,b,c}(\mathcal{S}_{a,b,c} + b/q) = \mathcal{S}_{a,b,c} + b/q$ and $(\mathcal{S}_{a,b,c} + b/q) \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) = \emptyset$. Hence the set $\mathcal{S}_{a,b,c} + b/q$ is invariant under the transformation $R_{a,b,c}$ and has empty intersection with the black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$ of the transformation $R_{a,b,c}$, which contradicts to the maximality of the set $\mathcal{S}_{a,b,c}$ established in Lemma 5.11.

(iii) Suppose on the contrary that both $[0, b/q)$ and $[-b/q, 0)$ are not contained in $\mathcal{S}_{a,b,c}$. Then

$$(5.97) \quad [c_0, c_0 + b/q) \subset \mathcal{S}_{a,b,c} \text{ and } [c_0 + a - b - b/q, c_0 + a - b) \in \mathcal{S}_{a,b,c}$$

by the first two conclusions of this lemma. First we show that there exists a nonnegative integer $1 \leq D \leq (2p - q)/(q - p)$ such that

$$(5.98) \quad (R_{a,b,c})^D([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) \neq \emptyset.$$

Suppose on the contrary that (5.98) does not hold. Then $(R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) \cap ([c_0 + a - b, c_0) + a\mathbb{Z}) = \emptyset$ for all $0 \leq n \leq (2p - q)/(q - p)$. Following the argument in (5.29), we have that $(R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z})$, $0 \leq n \leq (2p - q)/(q - p)$, are mutually disjoint. This together with $(R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) = (R_{a,b,c})^n([c - c_0, c + b - c_0 - a) \cap b\mathbb{Z}/q) + [0, b/q) + a\mathbb{Z}$, $0 \leq n \leq (2p - q)/(q - p)$, implies that $|\cup_{0 \leq n \leq (2p - q)/(q - p)} (R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) \cap ([0, a) \setminus [c_0 + a - b, c_0))| = \lfloor p/(q - p) \rfloor (q - p)/q > |[0, a) \setminus [c_0 + a - b, c_0)|$, which is a contradiction. This proves (5.98).

By (5.98), we may assume that the nonnegative integer D in (5.98) is the minimal integer such that (5.98) holds. Following the above

argument, we may conclude that

(5.99)

$(R_{a,b,c})^n([c - c_0, c + b - c_0 - a] + a\mathbb{Z}), 0 \leq n \leq D$, are mutually disjoint.

Now let us verify the following claim:

$$(5.100) \quad (R_{a,b,c})^n([c - c_0, c + b - c_0 - a] + a\mathbb{Z}) = [b_n + a - b, b_n] + a\mathbb{Z}$$

for some $b_n \in (0, a], 0 \leq n \leq D$, and

$$(5.101) \quad (R_{a,b,c})^D([c - c_0, c + b - c_0 - a] + a\mathbb{Z}) = [c_0 + a - b, c_0] + a\mathbb{Z}.$$

Proof of Claims (5.100) and (5.101). If $D = 0$, then (5.100) and (5.101) follow from (5.97) and the definition of the nonnegative integer D . Now we consider $D \geq 1$. Let $T_0 = [c - c_0, c + b - c_0 - a] + a\mathbb{Z}$ and define $T_n, 1 \leq n \leq D$, inductively by

(5.102)

$$T_n = \begin{cases} R_{a,b,c}(T_{n-1}) & \text{if } 0 \notin T_{n-1}, \\ R_{a,b,c}(T_{n-1}) \cup ([c - c_0, c + b - c_0 - a] + a\mathbb{Z}) & \text{if } 0 \in T_{n-1}. \end{cases}$$

Clearly $T_0 = [b_0 + a - b, b_0] + a\mathbb{Z}$ for some $b_0 \in (0, a]$. Inductively, we assume that $T_n = [\tilde{b}_n, b_n] + a\mathbb{Z}$ for some \tilde{b}_n, b_n with $b_n \in (0, a]$ and $b - a \leq b_n - \tilde{b}_n < a$. If $0 \notin T_n$, then either $[\tilde{b}_n, b_n] \subset [0, c_0 + a - b]$ or $[\tilde{b}_n, b_n] \subset [c_0, a]$. This implies that

$$(5.103) \quad \begin{aligned} T_{n+1} &= R_{a,b,c}(T_n) = [R_{a,b,c}(\tilde{b}_n), R_{a,b,c}(b_n) + b_n - \tilde{b}_n] + a\mathbb{Z} \\ &=: [\tilde{b}_{n+1}, b_{n+1}] + a\mathbb{Z} \end{aligned}$$

for some \tilde{b}_{n+1}, b_{n+1} with $b_{n+1} \in (0, a]$ and $b_{n+1} - \tilde{b}_{n+1} = b_n - \tilde{b}_n$. If $0 \in T_n$, then $\tilde{b}_n \leq 0$. Moreover $\tilde{b}_n \geq c_0 - a$ and $b_n \leq c_0 + a - b$, as otherwise T_n has nonempty intersection with the black hole $[c_0 + a - b, c_0] + a\mathbb{Z}$ of the transformation $R_{a,b,c}$, which contradicts to (5.98) and the observation that $T_n \subset \cup_{m=0}^n (R_{a,b,c})^m([c - c_0, c + b - c_0 - a] + a\mathbb{Z})$. Therefore

$$(5.104) \quad \begin{aligned} T_{n+1} &= R_{a,b,c}(T_n) \cup ([c - c_0, c + b - c_0 - a] + a\mathbb{Z}) \\ &= [\tilde{b}_n + \lfloor c/b \rfloor b, b_n + \lfloor c/b \rfloor b + b - a] + a\mathbb{Z} \\ &=: [\tilde{b}_{n+1}, b_{n+1}] + a\mathbb{Z} \end{aligned}$$

for some \tilde{b}_{n+1}, b_{n+1} with $b_{n+1} \in (0, a]$ and $b_{n+1} - \tilde{b}_{n+1} = b_n - \tilde{b}_n + b - a$. Combining (5.103) and (5.104) proceeds the inductive proof that

$$(5.105) \quad T_n = [\tilde{b}_n, b_n] + a\mathbb{Z} \text{ for all } 0 \leq n \leq D,$$

where $b_n \in (0, a]$ and $b_n - \tilde{b}_n \in [b - a, a]$. Observe that $(R_{a,b,c})^D([c - c_0, c + b - c_0 - a] + a\mathbb{Z}) \subset T_D \subset \cup_{n=0}^D (R_{a,b,c})^n([c - c_0, c + b - c_0 - a] + a\mathbb{Z})$. Then T_D has nonempty intersection with the black hole $[c_0 + a - b, c_0] + a\mathbb{Z}$ of the transformation $R_{a,b,c}$ by (5.99). This together with (5.105)

implies that either $[c_0 + a - b - b/q, c_0 + a - b) \subset T_D$, or $[c_0, c_0 + b/q) \subset T_D$ or $T_D = [c_0 + a - b, c_0) + a\mathbb{Z}$. Recall that $T_D \cap \mathcal{S}_{a,b,c} = \emptyset$ by Proposition 3.10. Then both $[c_0 + a - b - b/q, c_0 + a - b)$ and $[c_0, c_0 + b/q)$ have empty intersection with T_D by (5.97). Thus

$$(5.106) \quad T_D = [c_0 + a - b, c_0) + a\mathbb{Z}.$$

This together with (5.102), (5.103) and (5.104) implies that

$$(5.107) \quad \tilde{b}_n > 0 \text{ and } b_n - \tilde{b}_n = b - a \text{ for all } 0 \leq n \leq D.$$

The desired conclusions (5.100) and (5.101) then follow. \square

Let us return to the proof of the conclusion (iii). By (5.6), (5.97), (5.99), (5.100) and Proposition 3.10, either $[b_n + a - b, b_n) \subset [b/q, c_0 + a - b)$ or $[b_n + a - b, b_n) \subset [c_0 + b/q, a)$. This implies that

$$(5.108) \quad R_{a,b,c}(b_n + a - b - b/(2q)) + a\mathbb{Z} = b_{n+1} + a - b - b/(2q) + a\mathbb{Z}$$

for all $0 \leq n \leq D - 1$. By (5.97), (5.100), (5.101), (5.108), and Propositions 3.7 and 3.8, we have that

$$\begin{aligned} & (\tilde{R}_{a,b,c})^n(c_0 + a - b - b/(2q)) + a\mathbb{Z} \\ &= (\tilde{R}_{a,b,c})^n(b_D + a - b - b/(2q)) + a\mathbb{Z} \\ &= (\tilde{R}_{a,b,c})^{n-1}(b_{D-1} + a - b - b/(2q)) + a\mathbb{Z} = \dots \\ &= b_{D-n} + a - b - b/(2q) + a\mathbb{Z} \subset \mathcal{S}_{a,b,c}, \quad 0 \leq n \leq D. \end{aligned}$$

Hence $-b/(2q) + a\mathbb{Z} = \tilde{R}_{a,b,c}(c - c_0 - b/(2q)) + a\mathbb{Z} = (\tilde{R}_{a,b,c})^{D+1}(c_0 + a - b - b/(2q)) + a\mathbb{Z} \in \mathcal{S}_{a,b,c}$, which together with (5.6) implies that $[-b/q, 0) \in \mathcal{S}_{a,b,c}$. This is a contradiction. \square

5.3. Covering property of maximal invariant sets. In this subsection, we prove Theorems 5.6 and 5.7.

Proof of Theorem 5.6. Set $A_\lambda := \mathcal{S}_{a,b,c} \cap [0, c_0 + a - b) + \lambda + a\mathbb{Z}$ and $B_\lambda := \mathcal{S}_{a,b,c} \cap [c_0, a) + \lambda + a\mathbb{Z}$ for $\lambda \in b\mathbb{Z}$. We divide the proof into two cases.

Case 1: $a/b \notin \mathbb{Q}$.

Take $t_0 \in \mathcal{S}_{a,b,c}$. Then $(R_{a,b,c})^n(t_0) \in \mathcal{S}_{a,b,c}$ by Proposition 3.8. Write $(R_{a,b,c})^n(t_0) = t_0 + k_n b$, where the strictly increasing sequence $\{k_n\}_{n=0}^\infty$ of nonnegative integers is defined inductively by $k_0 = 0$ and

$$(5.109) \quad k_{n+1} - k_n = \begin{cases} \lfloor c/b \rfloor + 1 & \text{if } t_0 + k_n b \in [0, c_0 + a - b) + a\mathbb{Z} \\ \lfloor c/b \rfloor & \text{if } t_0 + k_n b \in [c_0, a) + a\mathbb{Z} \end{cases}$$

for $n \geq 0$. Then for any nonnegative integer l ,

$$(5.110) \quad \begin{aligned} t_0 + lb &= t_0 + k_n b + (l - k_n)b \\ &\in \left(\bigcup_{\lambda_2 \in [0, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}} B_{\lambda_2} \cup \left(\bigcup_{\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}} A_{\lambda_1} \right) \right) \end{aligned}$$

by (5.109), where k_n is so chosen that $k_n \leq l < k_{n+1}$. Therefore

$$\begin{aligned} & \{t_0 + lb - \lfloor (t_0 + lb)/a \rfloor a \mid 0 \leq l \in \mathbb{Z}\} \\ & \subset \left(\bigcup_{\lambda_2 \in [0, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}} B_{\lambda_2} \cap [0, a) \right) \cup \left(\bigcup_{\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}} A_{\lambda_1} \cap [0, a) \right) \end{aligned}$$

by (3.3) and (5.110). Notice that the left hand side of the above inclusion is a dense subset of $[0, a)$ by the assumption $a/b \notin \mathbb{Q}$, while its right hand side is the union of finitely many intervals that are right-open and left-closed by Theorem 5.4. Thus

$$\begin{aligned} [0, a) &= \left(\bigcup_{k=0}^{\lfloor c/b \rfloor - 1} (\mathcal{S}_{a,b,c} + kb) \cap [0, a) \right) \\ (5.111) \quad & \cup \left((\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b) + \lfloor c/b \rfloor b \cap [0, a) \right) \end{aligned}$$

and the conclusion (5.17) follows.

Case 2: $a/b = p/q$ and $c/b \in \mathbb{Z}/q$ for some coprime integers p and q .

Take $t_0 \in \mathcal{S}_{a,b,c} \cap b\mathbb{Z}/q$. The existence of such a point t_0 follows from (5.6) and the assumption that $\mathcal{S}_{a,b,c} \neq \emptyset$. Following the argument in (5.111), we have that

$$(5.112) \quad t_0 + lb \in \left(\bigcup_{\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}} A_{\lambda_1} \right) \cup \left(\bigcup_{\lambda_2 \in [0, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}} B_{\lambda_2} \right)$$

for all $0 \leq l \in \mathbb{Z}$. Observe that $\{t_0, t_0 + b, \dots, t_0 + (p-1)b\} + a\mathbb{Z} = b\mathbb{Z}/q$. The above observation together with (5.112) implies that

$$(5.113) \quad b\mathbb{Z}/q \subset \left(\bigcup_{\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}} A_{\lambda_1} \right) \cup \left(\bigcup_{\lambda_2 \in [0, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}} B_{\lambda_2} \right).$$

Combining (5.6) and (5.113) proves the desired covering property (5.17). \square

Proof of Theorem 5.7. (\implies) By Proposition 4.2 and the assumption that $\mathcal{D}_{a,b,c} = \emptyset$, we have that

$$(5.114) \quad t + \lambda \notin \mathcal{S}_{a,b,c} \text{ for all } t \in \mathcal{S}_{a,b,c} \cap [0, c_0 + a - b) \text{ and } \lambda \in [b, \lfloor c/b \rfloor b] \cap b\mathbb{Z};$$

and

$$(5.115) \quad t + \lambda \notin \mathcal{S}_{a,b,c} \text{ for all } t \in \mathcal{S}_{a,b,c} \cap [c_0, a) \text{ and } \lambda \in [b, \lfloor c/b \rfloor b - b] \cap b\mathbb{Z}.$$

Therefore the sets $\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b) + \lambda_1 + a\mathbb{Z}$, $\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}$, and $\mathcal{S}_{a,b,c} \cap [c_0, a) + \lambda_2 + a\mathbb{Z}$, $\lambda_2 \in [0, \lfloor c/b \rfloor b - b] \cap b\mathbb{Z}$, are mutually disjoint. This together with the covering property in Theorem 5.6 and

the periodic property (3.3) for the set $\mathcal{S}_{a,b,c}$ implies that

$$\begin{aligned}
a &= \sum_{\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}} |(\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b] + \lambda_1 + a\mathbb{Z}) \cap [0, a]| \\
&\quad + \sum_{\lambda_2 \in [0, \lfloor c/b \rfloor b - b] \cap b\mathbb{Z}} |(\mathcal{S}_{a,b,c} \cap [c_0, a] + \lambda_2 + a\mathbb{Z}) \cap [0, a]| \\
&= \sum_{\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}} |(\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b] + \lambda_1 + a\mathbb{Z}) \cap [\lambda_1, a + \lambda_1]| \\
&\quad + \sum_{\lambda_2 \in [0, \lfloor c/b \rfloor b - b] \cap b\mathbb{Z}} |(\mathcal{S}_{a,b,c} \cap [c_0, a] + \lambda_2 + a\mathbb{Z}) \cap [\lambda_2, a + \lambda_2]| \\
&= (\lfloor c/b \rfloor + 1)|\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b]| + \lfloor c/b \rfloor |\mathcal{S}_{a,b,c} \cap [c_0, a]|
\end{aligned}$$

and hence (5.18) follows.

(\Leftarrow) Set $A_\lambda = (\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b] + \lambda + a\mathbb{Z}) \cap [0, a]$ and $B_\lambda = (\mathcal{S}_{a,b,c} \cap [c_0, a] + \lambda + a\mathbb{Z}) \cap [0, a]$, $\lambda \in b\mathbb{Z}$. By Theorem 5.6, the sets A_{λ_1} , $\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}$ and B_{λ_2} , $\lambda_2 \in [0, \lfloor c/b \rfloor b - b] \cap b\mathbb{Z}$ form a covering for the interval $[0, a]$. This together with the assumption (5.18) and the periodic property (3.3) for the set $\mathcal{S}_{a,b,c}$ implies that

$$\begin{aligned}
a &= (\lfloor c/b \rfloor + 1)|\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b]| + \lfloor c/b \rfloor |\mathcal{S}_{a,b,c} \cap [c_0, a]| \\
&= \sum_{\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}} |A_{\lambda_1}| + \sum_{\lambda_2 \in [0, \lfloor c/b \rfloor b - b] \cap b\mathbb{Z}} |B_{\lambda_2}| \\
&\geq \left| \left(\bigcup_{\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}} A_{\lambda_1} \right) \cup \left(\bigcup_{\lambda_2 \in [0, \lfloor c/b \rfloor b - b] \cap b\mathbb{Z}} B_{\lambda_2} \right) \right| = a.
\end{aligned}$$

This implies that the intersection of any two of those sets A_{λ_1} , $\lambda_1 \in [0, \lfloor c/b \rfloor b] \cap b\mathbb{Z}$ and B_{λ_2} , $\lambda_2 \in [0, \lfloor c/b \rfloor b - b] \cap b\mathbb{Z}$, has zero Lebesgue measure. Hence they have empty intersection as those sets are finite union of intervals that are left-closed and right-open by Theorems 5.4 and 5.5. This together with Proposition 4.2 proves that $\mathcal{D}_{a,b,c} = \emptyset$. \square

5.4. Algebraic property of maximal invariant sets. In this subsection, we prove Theorem 5.8.

Proof of Theorem 5.8. (i): By (2.16) and (3.3), we have that

$$(5.116) \quad Y_{a,b,c}(t + a) = Y_{a,b,c}(t) + Y_{a,b,c}(a) \quad \text{for all } t \in \mathcal{S}_{a,b,c}.$$

Now we divide two cases to verify (5.19) for $t \in [0, a] \cap \mathcal{S}_{a,b,c}$.

Case 1: $a/b \notin \mathbb{Q}$.

For $t \in [0, c_0 + a - b) \cap \mathcal{S}_{a,b,c}$, we obtain from (2.13), (3.51), (2.16) and (3.65) that

$$\begin{aligned}
 Y_{a,b,c}(R_{a,b,c}(t)) &= |[0, R_{a,b,c}(t)) \cap \mathcal{S}_{a,b,c}| \\
 &= Y_{a,b,c}(R_{a,b,c}(0)) + |[R_{a,b,c}(0), R_{a,b,c}(t)) \cap \mathcal{S}_{a,b,c}| \\
 &= Y_{a,b,c}(R_{a,b,c}(0)) + |R_{a,b,c}([0, t) \cap \mathcal{S}_{a,b,c})| \\
 (5.117) \quad &= Y_{a,b,c}(R_{a,b,c}(0)) + Y_{a,b,c}(t).
 \end{aligned}$$

Similarly for $t \in [c_0, a) \cap \mathcal{S}_{a,b,c}$, we get

$$\begin{aligned}
 Y_{a,b,c}(R_{a,b,c}(t)) &= |[R_{a,b,c}(c_0), R_{a,b,c}(t)) \cap \mathcal{S}_{a,b,c}| + Y_{a,b,c}(c_0 + \lfloor c/b \rfloor b) \\
 &= |R_{a,b,c}([c_0, t) \cap \mathcal{S}_{a,b,c})| + |[0, c_0 + \lfloor c/b \rfloor b + a) \cap \mathcal{S}_{a,b,c}| \\
 &\quad - |[c_0 + \lfloor c/b \rfloor b, c_0 + \lfloor c/b \rfloor b + a) \cap \mathcal{S}_{a,b,c}| \\
 &= |[c_0, t) \cap \mathcal{S}_{a,b,c}| + |[0, \lfloor c/b \rfloor b + b) \cap \mathcal{S}_{a,b,c}| \\
 &\quad + |R_{a,b,c}([0, c_0 + a - b)) \cap \mathcal{S}_{a,b,c}| - Y_{a,b,c}(a) \\
 (5.118) \quad &= Y_{a,b,c}(t) + Y_{a,b,c}(R_{a,b,c}(0)) - Y_{a,b,c}(a).
 \end{aligned}$$

Then (5.19) follows from (5.117) and (5.118).

Case 2: $a/b = p/q$ and $c/b \in \mathbb{Z}/q$ for some coprime integers p and q .

For every $t \in [0, c_0 + a - b) \cap \mathcal{S}_{a,b,c}$, we obtain from (3.51) and (3.53) in Proposition 3.8, (5.14) in Theorem 5.5, and (5.116) that

$$\begin{aligned}
 Y_{a,b,c}(R_{a,b,c}(t)) &= Y_{a,b,c}(R_{a,b,c}(0)) + Y_{a,b,c}(t) \\
 (5.119) \quad &\in Y_{a,b,c}(c_1 + b - a) + Y_{a,b,c}(t) + Y_{a,b,c}(a)\mathbb{Z}.
 \end{aligned}$$

Similarly for every $t \in [c_0, a) \cap \mathcal{S}_{a,b,c}$, we have that

$$\begin{aligned}
 Y_{a,b,c}(R_{a,b,c}(t)) &= Y_{a,b,c}(t) + Y_{a,b,c}(R_{a,b,c}(0)) - Y_{a,b,c}(a) \\
 (5.120) \quad &\in Y_{a,b,c}(c_1 + b - a) + Y_{a,b,c}(t) + Y_{a,b,c}(a)\mathbb{Z}.
 \end{aligned}$$

Then (5.19) follows from (5.119) and (5.120).

(ii): Let $N_1, N_2, \delta, \delta'$ be as in Theorem 5.5. By (5.14), the set $\mathcal{K}_{a,b,c}$ of marks is given by

$$\mathcal{K}_{a,b,c} = \{Y_{a,b,c}((R_{a,b,c})^n(c - c_0 + b - a + \delta)) \mid 0 \leq n \leq N_1 + N_2\} + Y_{a,b,c}(a)\mathbb{Z}.$$

This, together with the first conclusion of this theorem and the fact that $[0, \delta)$ is contained in a gap, implies that

$$\begin{aligned}
 \mathcal{K}_{a,b,c} &= Y_{a,b,c}(R_{a,b,c}(\delta) - a) \\
 &\quad + \{nY_{a,b,c}(c_1 + b - a) \mid 0 \leq n \leq N_1 + N_2\} + Y_{a,b,c}(a)\mathbb{Z} \\
 (5.121) \quad &= \{nY_{a,b,c}(c_1 + b - a) \mid 1 \leq n \leq N_1 + N_2 + 1\} + Y_{a,b,c}(a)\mathbb{Z}.
 \end{aligned}$$

On the other hand, it follows from (5.14) and (5.15) that $(R_{a,b,c})^{N_1+N_2}(c-c_0+b+\delta) \in \delta + a\mathbb{Z}$. This together with the first conclusion of this theorem implies that

$$(5.122) \quad (N_1 + N_2 + 1)Y_{a,b,c}(c_1 + b - a) \in Y_{a,b,c}^d(a)\mathbb{Z}.$$

Combining (5.121) and (5.122) proves that $\mathcal{K}_{a,b,c}$ form a finite cyclic group generated by $Y_{a,b,c}(c_1 + b - a) + Y_{a,b,c}(a)\mathbb{Z}$.

(iii) By Theorem 5.4, there exists a nonnegative integer D such that $(R_{a,b,c})^n(c - c_0 + b - a) + [a - b, 0] + a\mathbb{Z}$, $0 \leq n \leq D$, are mutually disjoint, and

$$(5.123) \quad (R_{a,b,c})^D(c - c_0 + b - a) + [a - b, 0] + a\mathbb{Z} = [c_0 + a - b, c_0] + a\mathbb{Z}.$$

Therefore $(R_{a,b,c})^n(c - c_0 + b - a) \in \mathcal{S}_{a,b,c}$ for all $0 \leq n \leq D$, and

$$\begin{aligned} \mathcal{K}_{a,b,c} &= \cup_{n=0}^D \{Y_{a,b,c}((R_{a,b,c})^n(c - c_0 + b - a)) + Y_{a,b,c}(a)\mathbb{Z}\} \\ &= \cup_{n=0}^D \{(n+1)Y_{a,b,c}(c - c_0 + b - a) + Y_{a,b,c}(a)\mathbb{Z}\} \\ (5.124) \quad &= \cup_{m=1}^{D+1} \{mY_{a,b,c}(c_1 + b - a) + Y_{a,b,c}(a)\mathbb{Z}\} \end{aligned}$$

where the second equality follows from the first conclusion of this theorem. Moreover, it follows (5.123) that

$$(5.125) \quad (D+1)Y_{a,b,c}(c_1 + b - a) - Y_{a,b,c}(c_0) \in Y_{a,b,c}(a)\mathbb{Z}.$$

Also we notice that the nonnegative integer D satisfying (5.125) is unique as $(R_{a,b,c})^n(0) \notin a\mathbb{Z}$ for all positive integers n by the assumption $a/b \notin \mathbb{Q}$. This uniqueness, together with (5.124) and (5.125), proves the conclusion (iii). \square

Remark 5.13. Let N_1 and N_2 be as in Theorem 5.5. Notice that the mutually disjoint properties (iv) and (iv)' in Theorem 5.5, $nY_{a,b,c}(c_1 + b - a) \notin Y_{a,b,c}(a)\mathbb{Z}$ for all $0 \leq n \leq N_1 + N_2$. Thus $\mathcal{K}_{a,b,c}$ is a cyclic group of order $N_1 + N_2 + 1$.

6. MAXIMAL INVARIANT SETS WITH IRRATIONAL TIME-FREQUENCY LATTICE

In this section, we prove Theorem 2.4. To do so, we need characterize the non-triviality (2.10) of the maximal invariant set $\mathcal{S}_{a,b,c}$. By Theorems 5.4 and 5.5, after performing the holes-removal surgery, the maximal invariant set $\mathcal{S}_{a,b,c}$ becomes the real line with marks. This suggests that for the case that $a/b \notin \mathbb{Q}$ we can expand the line with marks by inserting holes $[0, b - a)$ at every location of marks to recover the maximal invariant set $\mathcal{S}_{a,b,c}$. Using the equivalence between the application of the piecewise linear transformation $R_{a,b,c}$ on the set $\mathcal{S}_{a,b,c}$ and a rotation on the circle with marks given in Theorem 5.8,

we can characterize the non-triviality (2.10) of the maximal invariant set $\mathcal{S}_{a,b,c}$ via two nonnegative integer parameters d_1 and d_2 for the case that $a/b \notin \mathbb{Q}$.

Theorem 6.1. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, $\lfloor c/b \rfloor \geq 2$, $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, $0 < c_1 := c - c_0 - \lfloor (c - c_0)/a \rfloor a < 2a - b$, and $a/b \notin \mathbb{Q}$. Then $\mathcal{S}_{a,b,c} \neq \emptyset$ if and only if there exist nonnegative integers d_1 and d_2 such that*

$$(6.1) \quad c - (d_1 + 1)(\lfloor c/b \rfloor + 1)(b - a) - (d_2 + 1)\lfloor c/b \rfloor(b - a) \in a\mathbb{Z},$$

$$(6.2) \quad \lfloor c/b \rfloor b + (d_1 + 1)(b - a) < c < \lfloor c/b \rfloor b + b - (d_2 + 1)(b - a),$$

and

$$(6.3) \quad \#E_{a,b,c} = d_1,$$

where $E_{a,b,c}$ is defined as in (2.19).

The nonnegative integers d_1 and d_2 in Theorem 6.1 satisfy $(d_1 + d_2 + 1) < a/(b - a)$ by (6.2), and they are uniquely determined by the triple (a, b, c) of positive numbers by (6.1) and the assumptions that $\lfloor c/b \rfloor \geq 2$ and $a/b \notin \mathbb{Q}$. We also notice that the nonnegative integer parameters d_1 and d_2 in Theorem 6.1 are indeed the numbers of holes contained in $[0, c_0 + a - b)$ and $[c_0, a)$ respectively.

In next two subsections, we prove Theorem 6.1, and apply Theorems 5.7 and 6.1 to prove Theorem 2.4 respectively.

6.1. Nontrivial maximal invariant sets with irrational time-frequency lattices. In this subsection, we prove Theorem 6.1.

Proof of Theorem 6.1. (\implies) Assume that $\mathcal{S}_{a,b,c} \neq \emptyset$. Let D be the nonnegative integer in Theorem 5.4. Then $(R_{a,b,c})^D([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) = [c_0 + a - b, c_0) + a\mathbb{Z}$, and the periodic holes $(R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z}) = (R_{a,b,c})^n(c - c_0) + [0, b - a) + a\mathbb{Z}$, $0 \leq n \leq D$, are mutually disjoint. This implies that

$$(6.4) \quad c_0 + a - b - (c - c_0 + d_1(\lfloor c/b \rfloor + 1)b + d_2\lfloor c/b \rfloor b) \in a\mathbb{Z},$$

where d_1, d_2 are the numbers of the indices $n \in [0, D - 1]$ such that $(R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z})$ is contained in $[0, c_0 + a - b) + a\mathbb{Z}$ and in $[c_0, a) + a\mathbb{Z}$ respectively. Then (6.1) follows from (6.4).

Observe that

$$D = d_1 + d_2$$

as for every $0 \leq n \leq D - 1$ the periodic hole $(R_{a,b,c})^n([c - c_0, c + b - c_0 - a) + a\mathbb{Z})$ is contained either in $[0, c_0 + a - b) + a\mathbb{Z}$ or in $[c_0, a) + a\mathbb{Z}$ by Theorem 5.4.

Notice that the periodic holes $(R_{a,b,c})^n([c-c_0, c+b-c_0-a]+a\mathbb{Z})$, $0 \leq n \leq d_1+d_2$, are mutually disjoint and have length $b-a$ by Theorem 5.4 and that there are d_1 (resp. d_2) holes contained in $[0, c_0+a-b)$ (resp. $[c_0, a)$) by the definition of integer parameters d_1 and d_2 . Therefore $d_1(b-a) < c_0+a-b$ and $d_2(b-a) < a-c_0$, which proves (6.2).

Let $\theta_{a,b,c} := Y_{a,b,c}(c-c_0+b)$ be as in Theorem 5.8, and $\tilde{\theta}_{a,b,c} = Y_{a,b,c}(c_1)$. Recall that there are d_1+d_2+1 holes contained in the interval $[0, a)$ by Theorem 5.4. This together with (5.116) implies that

$$(6.5) \quad Y_{a,b,c}(a) = a - (d_1 + d_2 + 1)(b - a)$$

and

$$(6.6) \quad \tilde{\theta}_{a,b,c} - Y_{a,b,c}(c-c_0) \in Y_{a,b,c}(a)\mathbb{Z},$$

where the inclusion holds because $[c-c_0, c+b-c_0-a)$ is a black hole of the transformation $\tilde{R}_{a,b,c}$, and

$$(6.7) \quad Y_{a,b,c}(c-c_0) - \theta_{a,b,c} = -Y_{a,b,c}(a).$$

From Theorem 5.8, we see that the marks are located at $Y_{a,b,c}(c-c_0) + n\theta_{a,b,c} + (a - (d_1 + d_2 + 1)(b - a))\mathbb{Z}$, $0 \leq n \leq d_1 + d_2$. This together with (6.5), (6.6) and (6.7) implies that the locations of marks are $n\tilde{\theta}_{a,b,c} + (a - (d_1 + d_2 + 1)(b - a))\mathbb{Z}$, $1 \leq n \leq d_1 + d_2 + 1$.

Recall that there are d_1 holes contained in $[0, c_0+a-b)$. Then $Y_{a,b,c}(c_0+a-b) = c_0+a-b-d_1(b-a)$, which implies that

$$(6.8) \quad c_0 - (d_1+1)(b-a) - (d_1+d_2+1)\tilde{\theta}_{a,b,c} \in (a - (d_1+d_2+1)(b-a))\mathbb{Z}.$$

Let m be the number of holes $(R_{a,b,c})^n([c-c_0, c+b-c_0-a)+a\mathbb{Z}) = (R_{a,b,c})^n([c_1, c_1+b-a)+a\mathbb{Z})$, $0 \leq n \leq d_1+d_2$, contained in $[0, c_1)+a\mathbb{Z}$. Due to the one-to-one correspondence between holes and marks, m is also the cardinality of the set $\{1 \leq n \leq d_1+d_2+1 \mid n\tilde{\theta}_{a,b,c} \in [0, \tilde{\theta}_{a,b,c}) + (a - (d_1+d_2+1)(b-a))\mathbb{Z}\}$. This, together with the observation that $[c_1, c_1+b-a)$ is the black hole of the transformation $\tilde{R}_{a,b,c}$, implies that

$$(6.9) \quad \tilde{\theta}_{a,b,c} = c_1 - m(b-a).$$

Let \tilde{m} be the unique integer such that $(d_1+d_2+1)\tilde{\theta}_{a,b,c} \in \tilde{m}(a - (d_1+d_2+1)(b-a)) + [0, a - (d_1+d_2+1)(b-a))$. We want to prove that

$$(6.10) \quad \tilde{m} = m.$$

For any $1 \leq l \leq \tilde{m}$, there exists one and only one $1 \leq n_l \leq d_1+d_2+1$ such that $n_l\tilde{\theta}_{a,b,c} \in l(a - (d_1+d_2+1)(b-a)) + [0, \tilde{\theta}_{a,b,c})$, which implies that $\tilde{m} \leq m$. Now we prove that $m \leq \tilde{m}$. Suppose on the contrary that $m > \tilde{m}$. Then there exists an integer $1 \leq n \leq d_1+d_2+1$ such that $n\tilde{\theta}_{a,b,c} \in [0, \tilde{\theta}_{a,b,c}) + (a - (d_1+d_2+1)(b-a))(\mathbb{Z} \setminus \{1, \dots, \tilde{m}\})$. This implies

that $n\tilde{\theta}_{a,b,c} \geq (\tilde{m}+1)(a - (d_1 + d_2 + 1)(b - a))$, which is a contradiction as $\tilde{\theta}_{a,b,c} \leq n\tilde{\theta}_{a,b,c} \leq (d_1 + d_2 + 1)\tilde{\theta}_{a,b,c} < (\tilde{m}+1)(a - (d_1 + d_2 + 1)(b - a))$ by the definition of the integer \tilde{m} , and hence (6.10) is established.

From (6.8), (6.9) and (6.10),

$$\begin{aligned} (d_1 + d_2 + 1)(c_1 - m(b - a)) &= (d_1 + d_2 + 1)\tilde{\theta}_{a,b,c} \\ &= c_0 - (d_1 + 1)(b - a) + m(a - (d_1 + d_2 + 1)(b - a)), \end{aligned}$$

which implies that

$$(6.11) \quad ma = (d_1 + d_2 + 1)c_1 - c_0 + (d_1 + 1)(b - a).$$

Then the condition (6.3) follows from (6.8), (6.9) and (6.11), and the definition of the integer d_1 .

(\Leftarrow) Let $c_1 = c - c_0 - \lfloor (c - c_0)/a \rfloor a$, and let d_1 and d_2 be as in (6.1) and (6.2). Then $c_1 \neq 0$ by $a/b \notin \mathbb{Q}$, and $-a < -c_0 + b - a \leq (d_1 + d_2 + 1)c_1 - c_0 + (d_1 + 1)(b - a) \leq (d_1 + d_2 + 1)c_1 < (d_1 + d_2 + 1)a$ by (6.1) and (6.2). Also from (6.1) and (6.2), we see that $(d_1 + d_2 + 1)c_1 - c_0 + (d_1 + 1)(b - a) \in (d_1 + d_2 + 1)\lfloor c/b \rfloor b - c + \lfloor c/b \rfloor b + (d_1 + 1)b + a\mathbb{Z} = a\mathbb{Z}$. Thus

$$(6.12) \quad (d_1 + d_2 + 1)c_1 - c_0 + (d_1 + 1)(b - a) = ma$$

for some integer $0 \leq m \leq d_1 + d_2$. Set

$$(6.13) \quad \tilde{\theta}_{a,b,c} = c_1 - m(b - a).$$

Then

$$(6.14) \quad (d_1 + d_2 + 1)\tilde{\theta}_{a,b,c} = c_0 - (d_1 + 1)(b - a) + m(a - (d_1 + d_2 + 1)(b - a))$$

by (6.12). This together with $0 \leq m \leq d_1 + d_2$ and $0 < c_0 - (d_1 + 1)(b - a) < a - (d_1 + d_2 + 1)(b - a)$ implies that

$$(6.15) \quad \tilde{\theta}_{a,b,c} \in (0, a - (d_1 + d_2 + 1)(b - a)).$$

We claim that

$$(6.16) \quad (n - n')\tilde{\theta}_{a,b,c} \notin (a - (d_1 + d_2 + 1)(b - a))\mathbb{Z}$$

for all $1 \leq n \neq n' \leq d_1 + d_2 + 1$. Suppose on the contrary that (6.16) are not true. Then $k\tilde{\theta}_{a,b,c} = l(a - (d_1 + d_2 + 1)(b - a))$ for some integers $l \in \mathbb{Z}$ and $k \in [1, d_1 + d_2] \cap \mathbb{Z}$. Then $k(m - \lfloor c/b \rfloor) = l(d_1 + d_2 + 1)$ and $k(m - \lfloor (\lfloor c/b \rfloor b/a) \rfloor) = l(d_1 + d_2 + 2)$ by the assumption $a/b \notin \mathbb{Q}$. Thus $l = k(\lfloor (\lfloor c/b \rfloor b/a) \rfloor - \lfloor c/b \rfloor)$, which is a contradiction as $1 \leq l < k$ by (6.15).

Denote $\mathcal{K}_{a,b,c} := \{n\tilde{\theta}_{a,b,c}\}_{n=1}^{d_1+d_2+1} + (a - (d_1 + d_2 + 1)(b - a))\mathbb{Z}$ and rewrite $\mathcal{K}_{a,b,c}$ as $\{z_n\}_{n=1}^{d_1+d_2+1} + (a - (d_1 + d_2 + 1)(b - a))\mathbb{Z}$ where $0 < z_1 < z_2 < \dots < z_{d_1+d_2+1} < a - (d_1 + d_2 + 1)(b - a)$. The existence of

such an increasing sequence $\{z_n\}_{n=1}^{d_1+d_2+1}$ follows from (6.16). Given any $\delta \in (0, c_0 - (d_1 + 1)(b - a))$ (respectively $\delta \in (c_0 - (d_1 + 1)(b - a), a - (d_1 + d_2 + 1)(b - a))$), it follows from (6.14) and (6.15) that for any integer $k \in [0, m]$ (resp. $k \in [0, m - 1]$) there is one and only one integer $n \in [1, d_1 + d_2 + 1]$ such that $n\tilde{\theta}_{a,b,c} \in k(a - (d_1 + d_2 + 1)(b - a)) + [\delta, \delta + \tilde{\theta}_{a,b,c})$ and for $k \in \mathbb{Z} \setminus [0, m]$ (resp. $k \in \mathbb{Z} \setminus [0, m - 1]$), and there is no integer $n \notin [1, d_1 + d_2 + 1]$ such that $n\tilde{\theta}_{a,b,c} \in k(a - (d_1 + d_2 + 1)(b - a)) + [\delta, \delta + \tilde{\theta}_{a,b,c})$. The above observations together with (6.15) and (6.16) imply that

$$(6.17) \quad \#([\delta, \delta + \tilde{\theta}_{a,b,c}) \cap (\{z_k\}_{k=1}^{d_1+d_2+1} + \{0, a - (d_1 + d_2 + 1)(b - a)\})) = m + 1$$

for $\delta \in (0, c_0 - (d_1 + 1)(b - a))$, and

$$(6.18) \quad \#([\delta, \delta + \tilde{\theta}_{a,b,c}) \cap (\{z_k\}_{k=1}^{d_1+d_2+1} + \{0, a - (d_1 + d_2 + 1)(b - a)\})) = m$$

for $\delta \in (c_0 - (d_1 + 1)(b - a), a - (d_1 + d_2 + 1)(b - a))$.

Now let us expand marks located at $\{z_l\}_{l=1}^{d_1+d_2+1} + (a - (d_1 + d_2 + 1)(b - a))\mathbb{Z}$ to holes of length $b - a$ located at $\{y_l\}_{l=1}^{d_1+d_2+1} + a\mathbb{Z}$ on the real line by

$$(6.19) \quad y_l = z_l + (l - 1)(b - a), 1 \leq l \leq d_1 + d_2 + 1.$$

Clearly $0 < y_1 < y_2 < \dots < y_{d_1+d_2+1} < a$. Now let us prove

$$(6.20) \quad (R_{a,b,c})^n(c - c_0) + a\mathbb{Z} = y_{l(n)} + a\mathbb{Z} \quad \text{for all } 0 \leq n \leq d_1 + d_2,$$

by induction on $0 \leq n \leq d_1 + d_2$, where $l(n) \in [1, d_1 + d_2 + 1]$ is the unique integer such that $z_{l(n)} \in (n + 1)\tilde{\theta}_{a,b,c} + (a - (d_1 + d_2 + 1)(b - a))\mathbb{Z}$. Applying (6.17) with sufficiently small δ and recalling $z_1 > 0$ proves that $z_{m+1} = \tilde{\theta}_{a,b,c}$. Combining (6.3) and (6.14) gives

$$(6.21) \quad z_{d_1+1} = c_0 + a - b - d_1(b - a) = (d_1 + d_2 + 1)\tilde{\theta}_{a,b,c} - m(a - (d_1 + d_2 + 1)(b - a)).$$

Thus

$$(6.22) \quad y_{l(0)} = y_{m+1} = z_{m+1} + m(b - a) = \tilde{\theta}_{a,b,c} + m(b - a) = c_1$$

and

$$(6.23) \quad y_{l(d_1+d_2)} = y_{d_1+1} = z_{d_1+1} + d_1(b - a) = c_0 + a - b.$$

The conclusion (6.20) for $n = 0$ follows from (6.22). Inductively we assume that (6.20) holds for $n = k \leq d_1 + d_2 - 1$. Then $z_{l(k)} \neq c_0 - (d_1 + 1)(b - a)$ by (6.14), (6.16) and the observation that $l(k) \neq d_1 + d_2 + 1$.

If $z_{l(k)} < c_0 - (d_1 + 1)(b - a)$, then $y_{l(k)} < c_0 + a - b$ by (6.23) and

$$\begin{aligned}
 (R_{a,b,c})^{k+1}(c - c_0) &= R_{a,b,c}((R_{a,b,c})^k(c - c_0)) \in R_{a,b,c}(y_{l(k)}) + a\mathbb{Z} \\
 &= y_{l(k)} + \lfloor c/b \rfloor b + b + a\mathbb{Z} \\
 (6.24) \quad &= z_{l(k)} + \tilde{\theta}_{a,b,c} + (m + l(k))(b - a) + a\mathbb{Z}.
 \end{aligned}$$

Note that $z_{l(k+1)} = z_{l(k)} + \tilde{\theta}_{a,b,c}$ or $z_{l(k+1)} = z_{l(k)} + \tilde{\theta}_{a,b,c} - (a - (d_1 + d_2 + 1)(b - a))$. For the first case, $l(k + 1) = l(k) + m + 1$ as $[z_{l(k)}, z_{l(k)} + \tilde{\theta}_{a,b,c}] \cap (\{z_k\}_{k=1}^{d_1+d_2+1} + \{0, a - (d_1 + d_2 + 1)(b - a)\}) = m + 1$ by (6.17) and hence

$$\begin{aligned}
 (R_{a,b,c})^{k+1}(c - c_0) &\in z_{l(k)} + \tilde{\theta}_{a,b,c} + (m + l(k))(b - a) + a\mathbb{Z} \\
 (6.25) \quad &= z_{l(k+1)} + (l(k + 1) - 1)(b - a) + a\mathbb{Z} = y_{l(k+1)} + a\mathbb{Z}.
 \end{aligned}$$

Similarly for the second case, $l(k + 1) = l(k) + m + 1 - (d_1 + d_2 + 1)$ since $\#([0, z_{l(k+1)}] + (a - (d_1 + d_2 + 1)(b - a)) \cap (\{z_k\}_{k=1}^{d_1+d_2+1} + \{0, a - (d_1 + d_2 + 1)(b - a)\})) = \#([0, z_{l(k)}] \cap \{z_k\}_{k=1}^{d_1+d_2+1}) \cup ([z_{l(k)}, z_{l(k)} + \tilde{\theta}_{a,b,c}] \cap \{z_k\}_{k=1}^{d_1+d_2+1}) = l(k) - 1 + m + 1 = m + l(k)$ by (6.17). Thus

$$\begin{aligned}
 (R_{a,b,c})^{k+1}(c - c_0) &\in z_{l(k)} + \tilde{\theta}_{a,b,c} + (m + l(k))(b - a) + a\mathbb{Z} \\
 &= z_{l(k+1)} + (a - (d_1 + d_2 + 1)(b - a)) \\
 &\quad + (l(k + 1) + (d_1 + d_2 + 1) - 1)(b - a) + a\mathbb{Z} \\
 (6.26) \quad &= y_{l(k+1)} + a\mathbb{Z}.
 \end{aligned}$$

This shows that the inductive conclusion holds when $z_{l(k)} < c_0 - (d_1 + 1)(b - a)$. Similarly we can show that the inductive conclusion holds when $z_{l(k)} > c_0 - (d_1 + 1)(b - a)$.

From (6.20), we see that for any $0 \leq n \leq d_1 + d_2 - 1$, $(R_{a,b,c})^n([c - c_0, c - c_0 + b - a]) + a\mathbb{Z} = [y_{l(n)}, y_{l(n)} + b - a] + a\mathbb{Z}$ is contained either in $[0, c_0 + a - b] + a\mathbb{Z}$ or $[c_0, a] + a\mathbb{Z}$, and $(R_{a,b,c})^D([c - c_0, c - c_0 + b - a]) + a\mathbb{Z} = [c_0 + a - b, c_0] + a\mathbb{Z}$. Therefore $\mathcal{S}_{a,b,c}$ is the complement of $\cup_{n=0}^{d_1+d_2} ([y_{l(n)}, y_{l(n)} + b - a] + a\mathbb{Z})$ and hence it is not an empty set. \square

6.2. Proof of Theorem 2.4. (XII): We observe that $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame if and only if $\mathcal{D}_{a,b,c} \neq \emptyset$ if and only if $\mathcal{S}_{a,b,c} \neq \emptyset$ and (5.18) does not hold if and only if the triple (a, b, c) satisfies (6.1), (6.2), (6.3), and $c - (\lfloor c/b \rfloor + 1)(d_1 + 1)(b - a) - \lfloor c/b \rfloor (d_1 + 1)(b - a) \neq a$. In the above argument, the first equivalence holds by Theorem 3.2, the second one follows from (3.27) and Theorem 5.7, and the last one is obtained from Theorem 6.1 and the observation that (5.18) holds if and only if $c - (\lfloor c/b \rfloor + 1)(d_1 + 1)(b - a) - \lfloor c/b \rfloor (d_2 + 1)(b - a) = a$ as there are d_1 holes of length $b - a$ in $\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b]$ and d_2 holes of length $b - a$ in $\mathcal{S}_{a,b,c} \cap [c_0, a]$ by Theorem 5.4.

7. MAXIMAL INVARIANT SETS WITH RATIONAL TIME-FREQUENCY LATTICE

In this section, we prove Theorem 2.5. To do so, let us consider how to expand the line with marks for the case that $a/b = p/q$ for some coprime integers p and q and $c/b \in \mathbb{Z}/q$. Unlike for the case that $b/a \notin \mathbb{Q}$, we insert gaps of either large or small sizes at each location of marks and also insert a gap at the origin, which makes the augmentation operation rather delicate and complicated. But on the other hand, we get the help from the finite cyclic group structure for the marks stated in Theorem 5.8.

Theorem 7.1. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c, b - a < c_0 := c - \lfloor c/b \rfloor b < a, 0 < c_1 := \lfloor c/b \rfloor b - \lfloor (\lfloor c/b \rfloor b/a) \rfloor a < 2a - b, \lfloor c/b \rfloor \geq 2, a/b = p/q$ for some coprime integers p and q , and $c/b \in \mathbb{Z}/q$. Then $\mathcal{S}_{a,b,c} \neq \emptyset$ if and only if the triple (a, b, c) of positive numbers is one of the following three types:*

- 1) $c_0 < \gcd(a, c_1)$.
- 2) $b - c_0 < \gcd(a, c_1 + b)$.
- 3) *There exist nonnegative integers d_1, d_2, d_3, d_4 such that*

$$(7.1) \quad 0 < B_d := a - (d_1 + d_2 + 1)(b - a) \in Nb\mathbb{Z}/q,$$

$$(7.2) \quad Nc_1 + (d_1 + d_3 + 1)(b - a) \in a\mathbb{Z},$$

$$(7.3) \quad (d_1 + d_2 + 1)(Nc_1 + (d_1 + d_3 + 1)(b - a)) - (d_1 + d_3 + 1)a \in Na\mathbb{Z},$$

$$(7.4) \quad c_0 = (d_1 + 1)(b - a) + (d_1 + d_3 + 1)B_d/N + \delta$$

for some $\delta \in (-\min(B_d/N, a - c_0), \min(B_d/N, c_0 + b - a))$,

$$(7.5) \quad \gcd(Nc_1 + (d_1 + d_3 + 1)(b - a), Na) = a,$$

and

$$(7.6) \quad \#E_{a,b,c}^d = d_1,$$

where $N = d_1 + d_2 + d_3 + d_4 + 2$ and $E_{a,b,c}^d$ is defined as in (2.20).

In Theorem 7.1, we insert a gap of large size at the origin for the first two cases, while a gap of small size is inserted at the origin for the third case. For the first two cases, no gaps of small size have been inserted at any location of marks and the size of gaps inserted is always c_0 for the first case and $b - c_0$ for the second case. For the third case, the nonnegative integer parameters d_1, d_2 are indeed the numbers of gaps of size $b - a + |\delta|$ inserted in $[0, c_0 + a - b)$ and $[c_0, a)$ respectively, and the nonnegative integer parameters d_3, d_4 are the numbers of gaps of

size $|\delta|$ inserted in $[0, c_0 + a - b)$ and $[c_0, a)$, excluding the one inserted at the origin, respectively.

In next two subsections, we give the proofs of Theorems 7.1 and 2.5.

7.1. Nontrivial maximal invariant sets with rational time-frequency

lattices. In this subsection, we prove Theorem 7.1. The necessity of Theorem 7.1 follows essentially from Theorems 5.5 and 5.8. We examine five cases to verify the sufficiency. For the case 1) $c_0 < \gcd(c_1, a)$, we show that $[c_0, \gcd(c_1, a)) + \gcd(c_1, a)\mathbb{Z}$ is an invariant set under the transformation $R_{a,b,c}$ and it has empty intersection with black holes of transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$. This together with the maximality of the set $\mathcal{S}_{a,b,c}$ in Lemma 5.11 implies that $\mathcal{S}_{a,b,c} \neq \emptyset$. Similarly for the case 2) $b - c_0 < \gcd(a, c_1 + b)$, we verify that $[0, \gcd(a, c_1 + b) - b + c_0) + \gcd(a, c_1 + b)\mathbb{Z}$ is invariant under the transformation $R_{a,b,c}$ and it has empty intersection with black holes of transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$. For the case 3), we start from putting marks at $h\mathbb{Z}$ and insert gaps $\frac{b-a+|\delta|}{2}(\text{sgn}(\delta + b/(2q)) + [-1, 1])$ at marks located at $lmh + NhZ$, $1 \leq l \leq d_1 + d_2 + 1$, and $\frac{|\delta|}{2}(\text{sgn}(\delta + b/(2q)) + [-1, 1])$ at other marked locations, where $N = d_1 + d_2 + d_3 + d_4 + 2$, $h = (a - (d_1 + d_2 + 1)(b - a))/N - |\delta|$ and $m = (Nc_1 + (d_1 + d_3 + 1)(b - a))/a$. We then show that the gaps just inserted form a set that is invariant under the transformation $R_{a,b,c}$ and that contains black holes of the transformations $\tilde{R}_{a,b,c}$ and $R_{a,b,c}$.

Proof of Theorem 7.1. (\implies) Let $N_1, N_2, \delta, \delta', h$ be as in Theorem 5.5.

Case 1: $\delta = c_0 + a - b$ and $\delta' = 0$.

In this case, $N_2 = 0$ and $(R_{a,b,c})^n([c_1, c_1 + c_0) + a\mathbb{Z}), 0 \leq n \leq N_1$, are mutually disjoint gap with $(R_{a,b,c})^{N_1}([c_1, c_1 + c_0) + a\mathbb{Z}) = [0, c_0) + a\mathbb{Z}$ by Theorem 5.5. Therefore $N_1 \neq 0$ as $c_1 > 0$. Observe that

$$(R_{a,b,c})^n([c_1, c_1 + c_0) + a\mathbb{Z}) = [c_1, c_1 + c_0) + n(c_1 - a) + a\mathbb{Z}, 0 \leq n \leq N_1$$

because $-a < c_1 - a < 0$ and $(R_{a,b,c})^n([c_1, c_1 + c_0) + a\mathbb{Z}) \subset [c_0, a) + a\mathbb{Z}$ for all $0 \leq n \leq N_1 - 1$. Replacing n by N_1 in the above equality and recalling that $(R_{a,b,c})^{N_1}([c_1, c_1 + c_0) + a\mathbb{Z}) = [0, c_0) + a\mathbb{Z}$ gives

$$c_1 + N_1(c_1 - a) \in a\mathbb{Z},$$

which implies that

$$(7.7) \quad c_1 = -N_1(c_1 - a) + ka$$

for some integer k . Write $c_1/b = r/q$ and let $m = \gcd(p, r)$. Then it follows from (7.7) that $N_1 + 1 \in p\mathbb{Z}/m$. This together with mutual

disjointness of the gaps $[c_1, c_1 + c_0) + n(c_1 - a) + a\mathbb{Z}, 0 \leq n \leq N_1$, implies that

$$(7.8) \quad N_1 + 1 = p/m,$$

as otherwise $p/m \leq N_1$ and $[c_1, c_1 + c_0) + a\mathbb{Z} = [c_1, c_1 + c_0) + (p/m)(c_1 - a) + a\mathbb{Z}$. Observe that

$$\cup_{n=0}^{N_1} (n(c_1 - a) + a\mathbb{Z}) = \cup_{n=0}^{p/m-1} (n(c_1 - a) + a\mathbb{Z}) = \{0, mb/q, \dots, (p-m)b/q\} + a\mathbb{Z}.$$

Therefore the mutual disjointness of the gaps $[c_1, c_1 + c_0) + n(c_1 - a) + a\mathbb{Z}, 0 \leq n \leq N_1$, becomes $c_0 \leq mb/q = \gcd(c_1, a)$. We notice that $\cup_{n=0}^{N_1} [c_1, c_1 + c_0) + n(c_1 - a) + a\mathbb{Z} = \cup_{n=0}^{p/m-1} ([0, mb/q) + mn b/q + a\mathbb{Z}) = \mathbb{R}$ if $c_0 = mb/q$. This proves the desired first condition $c_0 < \gcd(c_1, a)$ in Theorem 7.1.

Case 2: $\delta = 0$ and $\delta' = c_0 - a$.

In this case, $N_2 = 0$ and $(R_{a,b,c})^n([c_1 + c_0 - a, c_1 + b - a) + a\mathbb{Z}), 0 \leq n \leq N_1$, are mutually disjoint gap with $(R_{a,b,c})^{N_1}([c_1 + c_0 - a, c_1 + b - a) + a\mathbb{Z}) = [c_0 + a - b, a) + a\mathbb{Z}$ by Theorem 5.5. Therefore $N_1 \geq 1$ as $c_1 < 2a - b$. Observe that

$$(R_{a,b,c})^n([c_1 + c_0 - a, c_1 + b - a) + a\mathbb{Z}) = [c_1 + c_0 - a, c_1 + b - a) + n(c_1 + b - a) + a\mathbb{Z}$$

for all $0 \leq n \leq N_1$, because $0 < c_1 + b - a < a$ and $(R_{a,b,c})^n([c_1 + c_0 - a, c_1 + b - a) + a\mathbb{Z}) \subset [0, c_0 + a - b) + a\mathbb{Z}$ for all $0 \leq n \leq N_1 - 1$. Replacing n by N_1 in the above equality and recalling that $(R_{a,b,c})^{N_1}([c_1 + c_0 - a, c_1 + b - a) + a\mathbb{Z}) = [c_0 + a - b, a) + a\mathbb{Z}$ give

$$(7.9) \quad (N_1 + 1)(c_1 + b - a) \in a\mathbb{Z}.$$

This together with mutual disjointness of $[c_1 + c_0 - a, c_1 + b - a) + n(c_1 + b - a) + a\mathbb{Z}, 0 \leq n \leq N_1$, implies that

$$N_1 + 1 = a/\gcd(c_1 + b, a),$$

as otherwise $a/\gcd(c_1 + b, a) \leq N_1$ and $[c_1 + c_0 - a, c_1 + b - a) + a\mathbb{Z} = [c_1 + c_0 - a, c_1 + b - a) + (a/\gcd(c_1 + b, a))(c_1 + b - a) + a\mathbb{Z}$, which is a contradiction. Therefore mutual disjointness of the gaps $(R_{a,b,c})^n([c_1 + c_0 - a, c_1 + b - a) + a\mathbb{Z}), 0 \leq n \leq N_1$, becomes mutual disjointness of the gaps $[c_1 + c_0 - a, c_1 + b - a) + i\gcd(c_1 + b, a) + a\mathbb{Z}, 0 \leq i \leq a/\gcd(c_1 + b, a) - 1$, which holds if and only if $b - c_0 \leq \gcd(c_1 + b, a)$. Also we notice that

$$\begin{aligned} & \cup_{n=0}^{N_1} (R_{a,b,c})^n([c_1 + c_0 - a, c_1 + b - a) + a\mathbb{Z}) \\ &= \cup_{i=0}^{a/\gcd(c_1 + b, a) - 1} ([c_1 + c_0 - a, c_1 + b - a) + i\gcd(c_1 + b, a) + a\mathbb{Z}) \\ &= c_1 + b - a + [-\gcd(c_1 + b, a), 0) + \gcd(c_1 + b, a)\mathbb{Z} = \mathbb{R} \end{aligned}$$

if $b - c_0 = \gcd(c_1 + b, a)$, which contradicts to $\mathcal{S}_{a,b,c} \neq \emptyset$. This leads to the desired second condition $b - c_0 < \gcd(c_1 + b, a)$ in Theorem 7.1.

Case 3: $0 < \delta < c_0 + a - b$ and $\delta' = 0$.

By Theorem 5.5, $N_1 \geq 0$ and $N_2 \geq 1$. Denote by d_1, d_2 the number of big gaps $(R_{a,b,c})^n(c - c_0 + [0, b - a + \delta))$, $0 \leq n \leq N_1 - 1$, of length $b - a + \delta$ contained in $[0, c_0 + a - b - \delta) + a\mathbb{Z}$ and in $[c_0, a) + a\mathbb{Z}$ respectively, and similarly denote by d_3 and d_4 the number of small gaps $(R_{a,b,c})^m([c_0 + \lfloor c/b \rfloor b - \delta, c_0 + \lfloor c/b \rfloor b))$, $0 \leq m \leq N_2 - 1$, of length δ contained in $[\delta, c_0 + a - b - \delta) + a\mathbb{Z}$ and in $[c_0, a) + a\mathbb{Z}$ respectively. Now let us verify that (7.1)–(7.6) hold for the above nonnegative integer parameters d_1, d_2, d_3, d_4 .

Proof of (7.1). By Theorem 5.5, for every $0 \leq n \leq N_1 - 1$, the gap $(R_{a,b,c})^n(c - c_0 + [0, b - a + \delta) + a\mathbb{Z})$ is either contained in $[0, c_0 + a - b) + a\mathbb{Z}$ or $[c_0, a) + a\mathbb{Z}$. Hence

$$(7.10) \quad N_1 = d_1 + d_2.$$

Similarly

$$(7.11) \quad N_2 - 1 = d_3 + d_4$$

as for any $0 \leq m \leq N_2 - 1$, the gap $(R_{a,b,c})^m([c_0 + \lfloor c/b \rfloor b - \delta, c_0 + \lfloor c/b \rfloor b)) = (R_{a,b,c})^{m+1}([c_0 + a - b - \delta, c_0) \setminus [c_0 + a - b, c_0) + a\mathbb{Z})$ of length δ is contained either in $[0, c_0 + a - b - \delta) + a\mathbb{Z}$ and in $[c_0, a) + a\mathbb{Z}$. Combining (5.8), (5.14), (5.15), (7.10), and (7.11), we obtain that there are $(d_1 + d_2 + 1)$ gaps of length $b - a + \delta$ and $(d_3 + d_4 + 1)$ gaps of length δ , and $N := (d_1 + d_2 + d_3 + d_4 + 2)$ intervals of length h on one period $[0, a)$. Therefore

$$(7.12) \quad 0 < a - (d_1 + d_2 + 1)(b - a) = N(h + \delta) \in Nb\mathbb{Z}/q.$$

This proves (7.1).

Proof of (7.2). By (5.10), (5.12) and the definition of nonnegative integers d_i , $1 \leq i \leq 4$, we obtain that

$$(7.13) \quad c - c_0 + b - a + \delta + d_1(\lfloor c/b \rfloor + 1)b + d_2\lfloor c/b \rfloor b \in c_0 + a\mathbb{Z},$$

and

$$(7.14) \quad c_0 + \lfloor c/b \rfloor b - \delta + d_3(\lfloor c/b \rfloor + 1)b + d_4\lfloor c/b \rfloor b \in a\mathbb{Z}.$$

Adding (7.13) and (7.14) leads to $c + (d_1 + d_3 + 1)(\lfloor c/b \rfloor + 1)b + (d_2 + d_4)\lfloor c/b \rfloor b \in c_0 + a\mathbb{Z}$. Then $Nc_1 + (d_1 + d_3 + 1)(b - a) \in a\mathbb{Z}$ and (7.2) is true.

Proof of (7.4). By Theorem 5.5 and the definition of the integers d_1 and d_3 , the interval $[0, c_0 + a - b - \delta)$ is covered by d_1 gaps of length

$b - a + \delta$, $d_3 + 1$ gaps of length δ , and $d_1 + d_3 + 1$ intervals of length h . This together with (7.12) leads to

$$\begin{aligned} c_0 + a - b - \delta &= d_1(b - a + \delta) + (d_3 + 1)\delta + (d_1 + d_3 + 1)h \\ (7.15) \quad &= d_1(b - a) + (d_1 + d_3 + 1)B_d/N. \end{aligned}$$

This proves (7.4).

Proof of (7.3). Substituting the expression in (7.15) into (7.13), we obtain that

$$\begin{aligned} a\mathbb{Z} \ni c - c_0 + b - a + \delta - c_0 + d_1(\lfloor c/b \rfloor + 1)b + d_2\lfloor c/b \rfloor b - d_1a \\ &= d_1(\lfloor c/b \rfloor + 1)b + d_2\lfloor c/b \rfloor b + \lfloor c/b \rfloor b + b - (d_1 + 1)a \\ &\quad - (d_1 + 1)(b - a) - (d_1 + d_3 + 1)B_d/N \\ (7.16) \quad &= (d_1 + d_2 + 1)\lfloor c/b \rfloor b - (d_1 + d_3 + 1)B_d/N. \end{aligned}$$

Multiplying N at both sides of the above equation leads to the desired inclusion (7.3).

Proof of (7.5). Define

$$(7.17) \quad m = (Nc_1 + (d_1 + d_3 + 1)(b - a))/a.$$

Then m is a positive integer in $[1, N - 1]$ by (7.2) and the observation that $0 < Nc_1 + (d_1 + d_3 + 1)(b - a) < N(2a - b) + (d_1 + d_3 + 1)(b - a) = Na - (d_2 + d_4 + 1)(b - a) \leq Na$. Let $Y_{a,b,c}$ be as in Theorem 5.8 and let m_1 be the nonnegative integer in $[0, N - 1]$ such that $Y_{a,b,c}(c_1 + b - a) \in m_1h + Y_{a,b,c}(a)\mathbb{Z}$. We claim the following:

$$(7.18) \quad m_1 = m.$$

Recall that $(R_{a,b,c})^{N_1}([c_1, c_1 + b - a + \delta) + a\mathbb{Z}) = [c_0 + a - b - \delta, c_0) + a\mathbb{Z}$ by Theorem 5.5, and that there are $d_1 + d_3 + 1$ gaps in the interval $[0, c_0 + a - b - \delta)$. This together with Theorem 5.8 that

$$(7.19) \quad (d_1 + d_2 + 1)m_1h - (d_1 + d_3 + 1)h \in Y_{a,b,c}(a)\mathbb{Z} = Nh\mathbb{Z}.$$

Then the number of gaps of length $b - a + \delta$ contained $[0, c_1)$ is $((d_1 + d_2 + 1)m_1h - (d_1 + d_3 + 1)h)/Y_{a,b,c}(a) = ((d_1 + d_2 + 1)m_1 - (d_1 + d_3 + 1))/N$. This implies that there are m_1 gaps contained in $[0, c_1)$ with $((d_1 + d_2 + 1)m_1 - (d_1 + d_3 + 1))/N$ of them are gaps of length $b - a + \delta$. Hence

$$\begin{aligned} c_1 &= m_1h + \left(m_1 - \frac{(d_1 + d_2 + 1)m_1 - (d_1 + d_3 + 1)}{N}\right)\delta \\ &\quad + \frac{(d_1 + d_2 + 1)m_1 - (d_1 + d_3 + 1)}{N}(b - a + \delta) \\ &= (m_1a - (d_1 + d_3 + 1)(b - a))/N. \end{aligned}$$

This together with (7.17) proves (7.18).

We return to the proof of (7.5). The above claim (7.18), together with (7.10), (7.11), and the observation that $\mathcal{K}_{a,b,c}$ is a cyclic group generated by $Y_{a,b,c}(c_1 + b - a)$ and has order $N_1 + N_2 + 1 = N$ by Theorem 5.8 and Remark 5.13, proves (7.5).

Proof of (7.6). Applying (7.17) and (7.18), and recalling that $[c_1, c_1 + b - a)$ is a hole in the complement of the maximal invariant set $\mathcal{S}_{a,b,c}$, we have that

$$(7.20) \quad Y_{a,b,c}(c_1) - mh \in Nh\mathbb{Z}.$$

Then $n \in E_{a,b,c}^d$ if and only if $(R_{a,b,c})^n[c_1, c_1 + b - a + \delta)$ is a big gap contained in $[0, c_0 + a - b) + a\mathbb{Z}$. This implies that the cardinality of the set $E_{a,b,c}^d$ is equal to d_1 from the definition of the nonnegative integer d_1 .

This completes the proof of the necessity for the case that $\delta \in (0, c_0 + a - b)$ and $\delta' = 0$.

Case 4: $\delta' \in (c_0 - a, 0)$ and $\delta = 0$.

By Theorem 5.5, $N_1 \geq 0$ and $N_2 \geq 1$. Denote by d_1, d_2 the numbers of big gaps $(R_{a,b,c})^n(c - c_0 + [\delta', b - a]), 0 \leq n \leq N_1 - 1$, of length $b - a - \delta'$ contained in $[0, c_0 + a - b) + a\mathbb{Z}$ and in $[c_0 - \delta', a + \delta') + a\mathbb{Z}$ respectively, and similarly denote by d_3 and d_4 the numbers of small gaps $(R_{a,b,c})^m([c_0 + \lfloor c/b \rfloor b, c_0 + \lfloor c/b \rfloor b - \delta')), 0 \leq m \leq N_2 - 1$, of length $-\delta'$ contained in $[0, c_0 + a - b) + a\mathbb{Z}$ and in $[c_0 - \delta', a + \delta') + a\mathbb{Z}$ respectively. We may follow the argument for the first case and prove the desired properties (7.1)–(7.6) with the above nonnegative integers d_1, d_2, d_3 and d_4 .

Case 5: $\delta = \delta' = 0$.

By Theorem 5.5, $N_1 \geq 0$ and $N_2 \geq 1$. Denote by d_1, d_2 the numbers of gaps $(R_{a,b,c})^n(c - c_0 + [0, b - a]), 0 \leq n \leq N_1 - 1$, of length $b - a$ contained in $[0, c_0 + a - b) + a\mathbb{Z}$ and in $[c_0, a) + a\mathbb{Z}$ respectively, and similarly denote by d_3 and d_4 the numbers of $(R_{a,b,c})^m(c_0), 1 \leq m \leq N_2$, contained in $(0, c_0 + a - b) + a\mathbb{Z}$ or $[c_0, a) + a\mathbb{Z}$ respectively. Then we may follow the argument for the first case line by line and establish (7.1)–(7.6) with the above nonnegative integer parameters d_1, d_2, d_3 and d_4 .

This completes the proof of the necessity for the case that $\delta = \delta' = 0$ and also the proof of the necessity.

(\Leftarrow) We examine five cases to prove the sufficiency.

Case 1: $c_0 < \gcd(c_1, a)$.

Let $N + 1 = a/\gcd(c_1, a)$ for some nonnegative integer N and define $T = (\cup_{n=0}^N [c_0, \gcd(c_1, a)) + n(a - c_1)) + a\mathbb{Z}$. Then

$$\begin{aligned} T &= (\cup_{i=0}^N [c_0, \gcd(c_1, a)) + i\gcd(c_1, a)) + a\mathbb{Z} \\ (7.21) \quad &= [c_0, \gcd(c_1, a)) + \gcd(c_1, a)\mathbb{Z}, \end{aligned}$$

and T has empty intersection with black holes of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$, since $T \cap [0, c_0) = T \cap [c_1, c_0 + c_1) = \emptyset$, and for any $t \in T$,

$$R_{a,b,c}(t) = t + c_1 \in [c_0, \gcd(c_1, a)) + c_1 + \gcd(c_1, a)\mathbb{Z} = T.$$

Therefore $T \subset \mathcal{S}_{a,b,c}$ (in fact $T = \mathcal{S}_{a,b,c}$) as $\mathcal{S}_{a,b,c}$ is the maximal invariant set that has empty intersection with the black hole of the transformation $R_{a,b,c}$ by Lemma 5.11. Thus $\mathcal{S}_{a,b,c}$ is not an empty set as the restriction of the set T on $[0, a)$ consists of $a/\gcd(c_1, a)$ intervals of length $\gcd(c_1, a) - c_0 > 0$.

Case 2: $b - c_0 < \gcd(a, c_1 + b)$.

Write $a/\gcd(a, c_1 + b) = N + 1$ for some nonnegative integer N and define $T' = (\cup_{i=0}^N [0, \gcd(a, c_1 + b) - b + c_0) + i(c_1 + b - a)) + a\mathbb{Z} = [0, \gcd(a, c_1 + b) - b + c_0) + \gcd(a, c_1 + b)\mathbb{Z}$. Then T' has empty intersection with black holes of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$, since $T' \cap [c_1, c_1 + b - a) = T' \cap [c_0 + a - b, a) = \emptyset$, and for any $t \in T'$,

$$\begin{aligned} R_{a,b,c}(t) &= t + c_1 + b \in ([0, \gcd(a, c_1 + b) - b + c_0) \\ &\quad + c_1 + b + \gcd(a, c_1 + b)\mathbb{Z} = T'. \end{aligned}$$

Therefore $T' \subset \mathcal{S}_{a,b,c}$ (in fact $T' = \mathcal{S}_{a,b,c}$) as $\mathcal{S}_{a,b,c}$ is the maximal invariant set that has empty intersection with the black holes of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ by Lemma 5.11. Thus $\mathcal{S}_{a,b,c}$ is not an empty set as the restriction of the set T' on $[0, a)$ consists of $N + 1$ intervals of length $\gcd(a, c_1 + b) - b + c_0 > 0$.

Case 3: There exist nonnegative integers d_1, d_2, d_3, d_4 and $\delta \in (0, \min(B_d/N, c_0 + a - b))$ satisfying (7.1)–(7.6).

In this case, we set

$$(7.22) \quad h = (a - (d_1 + d_2 + 1)(b - a))/N - \delta,$$

$$(7.23) \quad m = (Nc_1 + (d_1 + d_3 + 1)(b - a))/a,$$

and

$$(7.24) \quad \tilde{m} = ((d_1 + d_2 + 1)m - (d_1 + d_3 + 1))/N.$$

Then

$$(7.25) \quad 0 < h \in b\mathbb{Z}/q$$

by (7.1) and (7.4); m is a positive integer no larger than $N - 1$, i.e.,

$$(7.26) \quad m \in \mathbb{Z} \cap [1, N - 1]$$

as $0 < Nc_1/a \leq m < (N(2a - b) + (d_1 + d_3 + 1)(b - a))/a < N$; and \tilde{m} is a nonnegative integer no larger than m ,

$$(7.27) \quad \tilde{m} \in [0, m] \cap \mathbb{Z}$$

by (7.3). Moreover,

$$(7.28) \quad \begin{aligned} & m \frac{a - (d_1 + d_2 + 1)(b - a)}{N} + \tilde{m}(b - a) \\ &= \frac{m}{N}a - \frac{d_1 + d_3 + 1}{N}(b - a) = c_1. \end{aligned}$$

In order to expand the real line \mathbb{R} with marks at $h\mathbb{Z}$ to create an invariant set under the transformation $R_{a,b,c}$, we insert gaps $[0, b - a + \delta)$ located at $lmh + Nh\mathbb{Z}$, $1 \leq l \leq d_1 + d_2 + 1$, and gaps $[0, \delta)$ otherwise. Recall from (7.5) that $(l - l')mh \notin Nh\mathbb{Z}$ for all $1 \leq l \neq l' \leq N$. Therefore we have inserted $d_1 + d_2 + 1$ gaps $[0, b - a + \delta)$ and $d_3 + d_4 + 1$ gaps $[0, \delta)$ on the interval $[0, Nh)$. Thus after performing the above expansion, the interval $[0, Nh)$ with marks on $[0, Nh) \cap h\mathbb{Z}$ becomes the interval $[0, Nh + (d_1 + d_2 + 1)(b - a + \delta) + (d_3 + d_4 + 1)\delta) = [0, a)$ with gaps $[y_i, y_i + h_i)$, $0 \leq i \leq N - 1$, where $0 = y_0 \leq y_1 \leq \dots \leq y_D$ and $h_i \in \{b - a + \delta, \delta\}$, $0 \leq i \leq N - 1$. Now we want to prove that

$$(7.29) \quad y_m = c_1.$$

For that purpose, we need the following claim:

Claim 7.2. *For $s \in [0, N - 1] \cap \mathbb{Z}$, the cardinality of the set $\{l \in [1, d_1 + d_2 + 1] \cap \mathbb{Z} \mid lmh \in sh + [0, mh) + Nh\mathbb{Z}\}$ is equal to $\tilde{m} + 1$ if $1 \leq s \leq d_1 + d_3 + 1$, and \tilde{m} otherwise.*

Proof. For any $i \in \mathbb{Z}$, let $k_i = \lfloor (iN + m + s - 1)/m \rfloor$ the unique integer such that $k_i mh \in [sh, sh + mh) + iNh$. Therefore $1 \leq k_i \leq d_1 + d_2 + 1$ if and only if $m \leq iN + m + s - 1 \leq (d_1 + d_2 + 1)m + m - 1$ if and only if $1 - s \leq iN \leq (d_1 + d_2 + 1)m - s = N\tilde{m} + (d_1 + d_3 + 1 - s)$. Therefore

$$\begin{aligned} & \#\{l \in [1, d_1 + d_2 + 1] \cap \mathbb{Z} \mid lmh \in sh + [0, mh) + Nh\mathbb{Z}\} \\ &= \sum_{i \in \mathbb{Z}} \#\{l \in [1, d_1 + d_2 + 1] \cap \mathbb{Z} \mid lmh \in sh + [0, mh) + iNh\} \\ &= \sum_{i \in \mathbb{Z}} \#\{k_i \in [1, d_1 + d_2 + 1] \cap \mathbb{Z}\} \\ &= \#\{[(1 - s)/N, \tilde{m} + (d_1 + d_3 + 1 - s)/N] \cap \mathbb{Z}\}. \end{aligned}$$

Counting the number of integers in the interval $[(1 - s)/N, \tilde{m} + (d_1 + d_3 + 1 - s)/N]$ proves the claim. \square

We return to the proof of the equality (7.29). By Claim 7.2, we have inserted \tilde{m} interval of length $b - a + \delta$ and $m - \tilde{m}$ interval of length δ in the marked interval $[0, mh)$. So after performing the expansion, the mark located at mh on the line becomes the gap located at $mh + (m - \tilde{m})\delta + \tilde{m}(b - a + \delta)$, which is equal to c_1 by (7.28). This completes the proof of the equality (7.29).

Next we show that

$$(7.30) \quad y_{d_1+d_3+1} = c_0 + a - b - \delta.$$

By (7.6), we have inserted d_1 gaps of length $b - a + \delta$ and $(d_1 + d_3 + 1) - d_1$ intervals of length δ in the marked interval $[0, (d_1 + d_3 + 1)h)$. Therefore the mark located at $(d_1 + d_3 + 1)h$ becomes $(d_1 + d_3 + 1)h + d_1(b - a + \delta) + (d_3 + 1)\delta = c_0 + a - b - \delta$ after inserting gaps, where the last equality follows from (7.4). Hence (7.30) follows.

Then we prove by induction on $0 \leq k \leq N - 1$ that

$$(7.31) \quad (R_{a,b,c})^k(c - c_0) + a\mathbb{Z} = y_{l(k)} + a\mathbb{Z}, \quad 0 \leq k \leq d_1 + d_2,$$

and

$$(7.32) \quad (R_{a,b,c})^m(c_0 + a - b - \delta) + a\mathbb{Z} = y_{l(m+d_1+d_2)} + a\mathbb{Z}, \quad 1 \leq m \leq d_3 + d_4 + 1,$$

where $l(k) \in (k + 1)m + N\mathbb{Z}$. We remark that $l(0) = m, l(d_1 + d_2 + d_3 + d_4 + 1) = l(N - 1) = 0$ and $l(d_1 + d_2) = d_1 + d_3 + 1$, where the last equality follows from (7.3).

Proof of (7.31) and (7.32). The conclusion (7.31) for $k = 0$ from (7.29) and the observation that $l(0) = m$. Inductively, we assume that the conclusion (7.31) holds for some $0 \leq k \leq d_1 + d_2 - 1$. Then $l(k) \neq 0, d_1 + d_3 + 1$ as $l(d_1 + d_2) = d_1 + d_3 + 1$ and $l(N - 1) = 0$. If $0 < l(k) < d_1 + d_3 + 1$, then

$$(7.33) \quad \begin{aligned} y_{l(k+1)} &= y_{l(k)} + (\tilde{m} + 1)(b - a + \delta) + (m - \tilde{m} - 1)\delta + mh \\ &= y_{l(k)} + c_1 + b - a \end{aligned}$$

if $l(k + 1) - l(k) = m$, and

$$(7.34) \quad \begin{aligned} y_{l(k+1)} + a &= y_{l(k)} + (\tilde{m} + 1)(b - a + \delta) + (m - \tilde{m} - 1)\delta + mh \\ &= y_{l(k)} + c_1 + b - a \end{aligned}$$

if $l(k + 1) - l(k) = m - N$, where (7.33) and (7.34) hold as we have inserted $\tilde{m} + 1$ gaps of size $b - a + \delta$ and $m - (\tilde{m} + 1)$ gaps of size δ on $[l(k)h, (l(k) + m)h)$ by Claim 7.2. Also we obtain from (7.30) that

$y_{l(k)} \in [0, c_0 + a - b - \delta)$ when $0 < l(k) < d_1 + d_3 + 1$, which together with the inductive hypothesis implies that

$$(7.35) \quad \begin{aligned} (R_{a,b,c})^{k+1}(c - c_0) + a\mathbb{Z} &= R_{a,b,c}(y_{l(k)}) + a\mathbb{Z} \\ &= y_{l(k)} + c_1 + b - a + a\mathbb{Z}. \end{aligned}$$

Combining (7.33), (7.34) and (7.35) leads to

$$(7.36) \quad (R_{a,b,c})^{k+1}(c - c_0) + a\mathbb{Z} = y_{l(k+1)} + a\mathbb{Z}$$

if $0 < l(k) < d_1 + d_3 + 1$.

Similarly if $d_1 + d_3 + 1 < l(k) \leq N - 1$, we have that

$$(7.37) \quad y_{l(k+1)} - y_{l(k)} \in c_1 + a\mathbb{Z}$$

because we have inserted \tilde{m} gaps of size $b - a + \delta$ and $m - \tilde{m}$ gaps of size δ on $[l(k)h, (l(k) + m)h)$ by Claim 7.2; and

$$(7.38) \quad (R_{a,b,c})^{k+1}(c - c_0) + a\mathbb{Z} = y_{l(k)} + c_1 + a\mathbb{Z},$$

since $y_{l(k)} \in [c_0, a)$ by (7.30). Combining (7.37) and (7.38) yields

$$(7.39) \quad (R_{a,b,c})^{k+1}(c - c_0) + a\mathbb{Z} = y_{l(k+1)} + a\mathbb{Z}$$

if $d_1 + d_3 + 1 < l(k) \leq N - 1$. Therefore we can proceed our inductive proof by (7.36) and (7.39). This completes the proof of the equalities in (7.31).

Notice that $y_{l(d_1+d_2)} = y_{d_1+d_3+1} = c_0 + a - b - \delta$ by (7.30). Hence

$$(7.40) \quad \begin{aligned} &(R_{a,b,c})^m(c_0 + a - b - \delta) + a\mathbb{Z} = (R_{a,b,c})^m(y_{l(d_1+d_2)}) + a\mathbb{Z} \\ &= (R_{a,b,c})^{m+d_1+d_2}(y_{l(0)}) + a\mathbb{Z} = (R_{a,b,c})^{m+d_1+d_2}(c - c_0) + a\mathbb{Z} \end{aligned}$$

for all $1 \leq m \leq d_3 + d_4 + 1$. Then we can follow the argument to prove (7.31) to show that (7.32) holds. \square

Finally from (7.31) and (7.32) the mutually disjoint gaps we have inserted are $(R_{a,b,c})^k(c - c_0) + [0, b - a + \delta) + a\mathbb{Z}$, $0 \leq k \leq d_1 + d_2$, and $(R_{a,b,c})^m(c_0 + a - b - \delta) + [0, \delta) + a\mathbb{Z}$, $1 \leq m \leq d_3 + d_4 + 1$. Moreover $(R_{a,b,c})^{d_1+d_2}(c - c_0) + [0, b - a + \delta) + a\mathbb{Z} = [c_0 + a - b - \delta, c_0) + a\mathbb{Z}$ by (7.30) and $l(d_1 + d_2) = d_1 + d_3 + 1$; and $(R_{a,b,c})^{d_3+d_4+1}(c_0 + a - b - \delta) + [0, \delta) + a\mathbb{Z} = (R_{a,b,c})^{N-1}(c_0 + a - b - \delta) + [0, \delta) + a\mathbb{Z} = [0, \delta) + a\mathbb{Z}$ by (7.40) and $l(N - 1) = 0$. Notice that the union of the above gaps is invariant under the transformation $R_{a,b,c}$ and contains the black holes of the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$. Therefore its complement is the set $\mathcal{S}_{a,b,c}$ by Lemma 5.11, whose restriction on $[0, a)$ has Lebesgue measure Nh . Thus the conclusion that $\mathcal{S}_{a,b,c} \neq \emptyset$ is established for this case.

Case 4: There exist nonnegative integers d_1, d_2, d_3, d_4 and $\delta \in (-\min(B_d/N, a - c_0), 0)$ satisfying (7.1)–(7.6).

Set $h = (a - (d_1 + d_2 + 1)(b - a))/N + \delta$ and $m = (Nc_1 + (d_1 + d_3 + 1)(b - a))/a$. We expand the real line \mathbb{R} with marks at $h\mathbb{Z}$ by inserting gaps $[\delta + a - b, 0)$ located at $lmh + Nh\mathbb{Z}$, $1 \leq l \leq d_1 + d_2 + 1$, and gaps $[\delta, 0)$ otherwise. After performing the above augmentation operation, the interval $[0, Nh)$ with marks $[0, Nh) \cap h\mathbb{Z}$ becomes the interval $[0, a)$ with gaps $[y_i + h_i, y_i)$, $0 \leq i \leq N_1$, where $0 < y_1 \leq \dots \leq y_N = a$ and $h_i \in \{\delta + a - b, \delta\}$, $1 \leq i \leq N$. We follow the argument used in Case 3 to show that $y_m = c_1 + b - a$, $y_{d_1+d_3+1} = c_0 - \delta$ and for $0 \leq k \leq N - 1$,

$$(7.41) \quad (R_{a,b,c})^k(c - c_0 + b) + a\mathbb{Z} = y_{l(k)} + a\mathbb{Z}, \quad 0 \leq k \leq d_1 + d_2,$$

and

$$(7.42) \quad (R_{a,b,c})^m(c_0 - \delta) + a\mathbb{Z} = y_{l(k)} + a\mathbb{Z}, \quad 1 \leq m \leq d_3 + d_4 + 1,$$

where $l(k) \in (k + 1)m + N\mathbb{Z}$. Therefore $\mathcal{S}_{a,b,c}$ is the complement of $(\cup_{n=0}^{d_1+d_2} [y_{l(k)} + a - b + \delta, y_{l(k)} + a\mathbb{Z}) \cup (\cup_{m=1}^{d_3+d_4+1} [y_{l(k)} + \delta, y_{l(k)} + a\mathbb{Z}))$, whose restriction on $[0, a)$ has Lebesgue measure $Nh > 0$. This prove the sufficiency for Case 4.

Case 5: There exist nonnegative integers d_1, d_2, d_3, d_4 and $\delta = 0$ satisfying (7.1)–(7.6).

In this case, we set $h = (a - (d_1 + d_2 + 1)(b - a))/N$ and $m = (Nc_1 + (d_1 + d_3 + 1)(b - a))/a$, and expand the real line \mathbb{R} with marks at $h\mathbb{Z}$ by inserting gaps $[0, b - a)$ located at $lmh + Nh\mathbb{Z}$, $1 \leq l \leq d_1 + d_2 + 1$, and gaps of zero length otherwise, c.f. the fourth subfigure of Figure 2 in Section 2. Then after performing the above operation, the interval $[0, Nh)$ becomes the interval $[0, a)$ with gaps $[y_i, y_i + h_i)$, $0 \leq i \leq N - 1$, where $0 = y_0 \leq y_1 \leq \dots \leq y_{N-1}$ and $h_i \in \{b - a, 0\}$, $0 \leq i \leq N - 1$. We follow the argument in Case 3 to show that $y_m = c_1$, $y_{d_1+d_3} = c_0 + a - b$ and by induction on $0 \leq k \leq N - 1$ that

$$(7.43) \quad (R_{a,b,c})^k(c - c_0) + a\mathbb{Z} = y_{l(k)} + a\mathbb{Z}, \quad 0 \leq k \leq d_1 + d_2,$$

and

$$(7.44) \quad (R_{a,b,c})^m(c_0) + a\mathbb{Z} = y_{l(m+d_1+d_2)} + a\mathbb{Z}, \quad 1 \leq m \leq d_3 + d_4 + 1,$$

where $l(k) \in (k + 1)m + N\mathbb{Z}$. Thus the union of gaps of size $b - a$ is $\cup_{n=0}^{d_1+d_2} (R_{a,b,c})^n([c - c_0, c - c_0 + b - a) + a\mathbb{Z})$ with $(R_{a,b,c})^{d_1+d_2}([c - c_0, c - c_0 + b - a) + a\mathbb{Z}) = [c_0 + a - b, c_0) + a\mathbb{Z}$. Therefore $\mathcal{S}_{a,b,c}$ is the complement of the above union of finite gaps and the sufficiency in the fifth case follows. \square

7.2. Proof of Theorem 2.5. (XIII): By Theorem 3.2, $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame if and only if $\mathcal{D}_{a,b,c} \neq \emptyset$, which in turn becomes $\mathcal{S}_{a,b,c} \neq \emptyset$ and $(\lfloor c/b \rfloor + 1)|\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b]| + \lfloor c/b \rfloor |\mathcal{S}_{a,b,c} \cap [c_0, a]| \neq a$ by (3.27) and Theorem 5.7. For the case that the triple (a, b, c) satisfies the first condition in Theorem 7.1, it follows from the argument used in the proof of Theorem 7.1 that $\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b) = \emptyset$ and $\mathcal{S}_{a,b,c} \cap [c_0, a) = \cup_{i=0}^N [c_0, \gcd(a, c_1)) + i\gcd(a, c_1)$, where $N + 1 = a/\gcd(a, c_1)$. Hence (5.18) holds if and only if $(N + 1)\lfloor c/b \rfloor(\gcd(a, c_1) - c_0) = a$ if and only if $\lfloor c/b \rfloor(\gcd(a, c_1) - c_0) = \gcd(a, c_1)$. For the case that the triple (a, b, c) satisfies the second condition in Theorem 7.1, $\mathcal{S}_{a,b,c} \cap [c_0, a) = \emptyset$ and $\mathcal{S}_{a,b,c} \cap [0, c_0 + a - b) = \cup_{i=0}^N [0, \gcd(c_1 + b, a) - b + c_0) + i\gcd(c_1 + b, a)$, where $N = a/\gcd(c_1 + b, a) - 1$. Hence (5.18) holds if and only if $(N + 1)(\lfloor c/b \rfloor + 1)(\gcd(c_1 + b, a) - b + c_0) = a$ if and only if $(\lfloor c/b \rfloor + 1)(\gcd(c_1 + b, a) - b + c_0) = \gcd(c_1 + b, a)$. For the case that the triple (a, b, c) satisfies the third condition in Theorem 7.1, there are $d_1 + d_3 + 1$ intervals of length h contained in $[0, c_0 + b - a)$ and $d_2 + d_4 + 1$ intervals of length h contained in $[c_0, a)$, where $h + |\delta| = B_d/N$. Therefore (5.18) holds if and only if $(\lfloor c/b \rfloor + 1)(d_1 + d_3 + 1)h + \lfloor c/b \rfloor(d_2 + d_4 + 1)h = a$ if and only if $h = a/(\lfloor c/b \rfloor N + (d_1 + d_3 + 1))$ if and only if $a/(\lfloor c/b \rfloor N + (d_1 + d_3 + 1)) + |\delta| = B_d/N$. Therefore the conclusion (XIII) holds by Theorem 7.1.

(XIV): This conclusion is given in [25, Section 3.3.6.1].

APPENDIX A. ALGORITHM

In this appendix, we provide a finite-step process to verify whether the Gabor system $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame for any given triple of (a, b, c) of positive numbers.

By Theorems 2.1–2.5, it suffices to consider triples satisfying assumptions in Conclusion (XII) of Theorem 2.4 and in Conclusion (XIII) of Theorem 2.5. For a triple (a, b, c) of positive numbers satisfying assumptions in Conclusion (XII) of Theorem 2.4, we let $A_0 = [c_1, c_1 + b - a)$ (the black hole of the piecewise linear transformation $\tilde{R}_{a,b,c}$ in (2.14)) and $S_0 = [0, a) \setminus A_0$, and define holes $A_k := [\alpha_k, \beta_k) \subset [0, a)$ and invariant sets $S_k \subset [0, a)$, $k \geq 1$, iteratively

$$A_k = \begin{cases} R_{a,b,c}(\alpha_{k-1}) - \lfloor (R_{a,b,c}(\alpha_{k-1})/a) \rfloor a + [0, b - a) & \text{if } A_{k-1} \subset [0, c_0 + a - b) \text{ or } A_{k-1} \subset [c_0, a), \\ A_{k-1} & \text{if } A_{k-1} = [c_0 + a - b, c_0), \\ [0, a) & \text{otherwise,} \end{cases}$$

and $S_k = S_{k-1} \setminus A_k$, $k \geq 1$. By Theorem 5.4, there exists a nonnegative integer $L \leq a/(b - a)$ such that A_k is invariant for $k \geq L$, which implies

that S_L is the maximal invariant set $\mathcal{S}_{a,b,c}$ in (2.9) and $S_k = S_L$ for all $k \geq L$. Thus by Theorems 3.2 and 5.7, $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if either $S_L = \emptyset$ or $(\lfloor c/b \rfloor + 1)|S_L \cap [0, c_0 + a - b]| + \lfloor c/b \rfloor |S_L \cap [c_0, a]| = a$.

For a triple (a, b, c) of positive numbers that satisfies assumptions in Conclusion (XIII) of Theorem 2.5, we let $B_0 = [c_1, c_1 + b - a)$ and $T_0 = [0, a) \setminus B_0$, and define holes $B_k := [\gamma_k, \delta_k) \subset [0, a)$ and invariant sets $T_k \subset [0, a)$, $k \geq 1$, iteratively

$$B_k := \begin{cases} R_{a,b,c}(\gamma_{k-1}) - \lfloor (R_{a,b,c}(\gamma_{k-1})/a) \rfloor a + [0, \delta_{k-1} - \gamma_{k-1}) & \text{if } 0 \leq \gamma_{k-1} < \delta_{k-1} \leq c_0 + a - b, \\ R_{a,b,c}(\gamma_{k-1}) - \lfloor (R_{a,b,c}(\gamma_{k-1})/a) \rfloor a + [0, c_0 + a - b - \gamma_{k-1}) & \text{if } 0 \leq \gamma_{k-1} < c_0 + a - b < \delta_{k-1} \leq c_0, \\ B_{k-1} & \text{if } c_0 + a - b \leq \gamma_{k-1} < \delta_{k-1} \leq c_0, \\ c - \lfloor c/a \rfloor a + [0, \delta_{k-1} - c_0) & \text{if } c_0 + b - a \leq \gamma_{k-1} < c_0 < \delta_{k-1} \leq a, \\ R_{a,b,c}(\gamma_{k-1}) - \lfloor (R_{a,b,c}(\gamma_{k-1})/a) \rfloor a + [0, \delta_{k-1} - \gamma_{k-1}) & \text{if } c_0 \leq \gamma_{k-1} < \delta_{k-1} \leq a, \\ [0, a) & \text{otherwise,} \end{cases}$$

and $T_k = T_{k-1} \setminus B_k$, $k \geq 1$. By (5.16) and Theorem 5.5, there exists a nonnegative integer L such that $B_k \subset [c_0 + a - b, c_0)$ or $B_k = [0, a)$ for all $k \geq L$, which implies that T_L is the maximal invariant set $\mathcal{S}_{a,b,c}$ in (2.9). Therefore by Proposition 4.2 and Theorems 3.2 and 5.7, $\mathcal{G}(\chi_{[0,c]}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if either $T_L = \emptyset$ or $(\lfloor c/b \rfloor + 1)|T_L \cap [0, c_0 + a - b]| + \lfloor c/b \rfloor |T_L \cap [c_0, a]| = a$.

APPENDIX B. SAMPLING SIGNALS IN A SHIFT-INVARIANT SPACE

An interesting problem in sampling is to identify generators ϕ and sampling-shift lattices $a\mathbb{Z} \times b\mathbb{Z}$ such that any signal f in the shift-invariant space

$$(B.1) \quad V_2(\phi, b\mathbb{Z}) := \left\{ \sum_{\lambda \in b\mathbb{Z}} d(\lambda) \phi(t - \lambda) : \sum_{\lambda \in b\mathbb{Z}} |d(\lambda)|^2 < \infty \right\}$$

can be stably recovered from its equally-spaced samples $f(t_0 + \mu)$, $\mu \in a\mathbb{Z}$, for any initial sampling position t_0 , that is, there exist positive constants A and B such that

$$(B.2) \quad A\|f\|_2 \leq \left(\sum_{\mu \in a\mathbb{Z}} |f(t_0 + \mu)|^2 \right)^{1/2} \leq B\|f\|_2$$

for all $f \in V_2(\phi, b\mathbb{Z})$ and $t_0 \in \mathbb{R}$. The range of sampling-shift parameters a and b is fully known only for few generators ϕ . For instance, the classical Whittaker-Shannon-Kotel'nikov sampling theorem states that

(B.2) hold for signals bandlimited to $[-\sigma, \sigma]$ if and only if $a \leq b = \pi/\sigma$. For the uniform spline generator $\underbrace{\chi_{[0,b)} * \cdots * \chi_{[0,b)}}_{n \text{ times}}$ obtained by convo-

luting the characteristic function on $[0, b)$ for n times, (B.2) hold if and only if $a < b$ [1, 36]. In this appendix, we consider the above range problem for the generator χ_I , the characteristic function on an interval I . Our results indicate that it is almost equivalent to the abc -problem for Gabor systems, and hence geometry of the range of sampling-shift parameters could be arbitrary complicated.

We say that $\{\phi(\cdot - \lambda) : \lambda \in b\mathbb{Z}\}$ is a *Riesz basis* for the shift-invariant space $V_2(\phi, b\mathbb{Z})$ if there exist positive constants A and B such that

$$(B.3) \quad A \left(\sum_{\lambda \in b\mathbb{Z}} |d(\lambda)|^2 \right)^{1/2} \leq \left\| \sum_{\lambda \in b\mathbb{Z}} d(\lambda) \phi(\cdot - \lambda) \right\|_2 \leq B \left(\sum_{\lambda \in b\mathbb{Z}} |d(\lambda)|^2 \right)^{1/2}$$

for all square-summable sequences $(d(\lambda))_{\lambda \in b\mathbb{Z}}$. For an interval $I = [d, c+d)$, $\{\chi_I(\cdot - \lambda) : \lambda \in b\mathbb{Z}\}$ is a Riesz basis for the shift-invariant space $V_2(\chi_I, b\mathbb{Z})$ except that $2 \leq c/b \in \mathbb{Z}$. Therefore except that $2 \leq c/b \in \mathbb{Z}$, any signal f in $V_2(\chi_I, b\mathbb{Z})$ can be stably recovered from its equally-spaced samples $f(t_0 + \mu)$, $\mu \in a\mathbb{Z}$, for any initial sampling position t_0 if and only if infinite matrices $\mathbf{M}_{a,b,c}(t)$, $t \in \mathbb{R}$, in (2.1) have the uniform stability property (3.4), c.f. [2, 37, 39]. This together with the characterization of frame property of the Gabor system $\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z}/b)$ in [32] leads to the following equivalence between our sampling problem associated with the box generator χ_I and the abc -problem for Gabor systems.

Proposition B.1. *Let $a, b > 0$ and I be an interval with length $c > 0$. Except that $2 \leq c/b \in \mathbb{Z}$, the following two statements are equivalent.*

- (i) *Any signal f in the shift-invariant space $V_2(\chi_I, b\mathbb{Z})$ can be stably recovered from equally-spaced samples $f(t_0 + \mu)$, $\mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$.*
- (ii) *$\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame for $L^2(\mathbb{R})$.*

If $I = [d, c+d)$ with $2 \leq c/b \in \mathbb{Z}$, then the shift-invariant space $V_2(\chi_I, b\mathbb{Z})$ is not closed in $L^2(\mathbb{R})$, but its closure is the shift-invariant space generated by $\chi_{I'}$ where $I' = [d, d+b)$. Therefore for the case that $I = [d, c+d)$ with $2 \leq c/b \in \mathbb{Z}$, any signal f in $V_2(\chi_I, b\mathbb{Z})$ can be stably recovered from equally-spaced samples $f(t_0 + \mu)$, $\mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$ if and only if any signal f in $V_2(\chi_{[d, d+b)}, b\mathbb{Z})$ can be stably recovered from equally-spaced samples $f(t_0 + \mu)$, $\mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$ if and only if $a \leq b$. This together with Theorems 2.1–2.5 and Proposition B.1 provides the full

classification of sampling-shift lattices $a\mathbb{Z} \times b\mathbb{Z}$ such that any signal f in $V_2(\chi_I, b\mathbb{Z})$ can be stably recovered from equally-spaced samples $f(t_0 + \mu)$, $\mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$.

We remark that two statements in Proposition B.1 are not equivalent for the case that $2 \leq c/b \in \mathbb{Z}$ and $a \leq b$. The reason is that under that assumption on the triple (a, b, c) , $\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame by Theorems 2.1 and 2.2, while any signal f in $V_2(\chi_I, b\mathbb{Z})$ can be stably recovered from equally-spaced samples $f(t_0 + \mu)$, $\mu \in a\mathbb{Z}$, for any initial sampling position $t_0 \in \mathbb{R}$.

Oversampling, i.e., $a < b$, helps for perfect reconstruction of band-limited signals and spline signals from their equally-spaced samples [1, 2, 37]. Our results indicate that oversampling does not always implies the stability of sampling and reconstruction process for signals in the shift-invariant space $V_2(\chi_I, b\mathbb{Z})$.

APPENDIX C. NON-ERGODICITY OF A PIECEWISE LINEAR TRANSFORMATION

Recall that the piecewise linear transformation $R_{a,b,c}$ is non-expansive and non-measure-preserving map on the real line. In this appendix, we study non-ergodicity associated with this new map on the line. Particularly, we establish the equivalence between the empty set property for the maximal invariant set $\mathcal{S}_{a,b,c}$ and non-ergodicity of the piecewise linear transformation $R_{a,b,c}$. The reader may refer to [38] for ergodic theory of various dynamic systems.

Theorem C.1. *Let (a, b, c) be a triple of positive numbers satisfying $a < b < c$, $b - a < c_0 := c - \lfloor c/b \rfloor b < a$, $0 < c_1 := \lfloor c/b \rfloor b - \lfloor (\lfloor c/b \rfloor b / a) \rfloor a < 2a - b$ and $\lfloor c/b \rfloor \geq 2$, and let $R_{a,b,c}$ and $\mathcal{S}_{a,b,c}$ be the piecewise linear transformation in (2.13) and its maximal invariant set in (2.9) respectively. Then $\mathcal{S}_{a,b,c} = \emptyset$ if and only if for any $t \in \mathbb{R}$ there exists $t_0 \in [c_0 + a - b, c_0)$ such that*

$$(C.1) \quad \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} f((R_{a,b,c})^k(t))}{n} = f(t_0)$$

for all continuous periodic functions f with period a .

Proof. (i) \implies (ii) Clearly it suffices to prove that for any $t \in \mathbb{R}$ there exists a nonnegative integer N such that

$$(C.2) \quad (R_{a,b,c})^N(t) \in [c_0 + a - b, c_0) + a\mathbb{Z}.$$

Suppose, on the contrary, that such a nonnegative integer N does not exist. Then $(R_{a,b,c})^n(t) \notin [c_0 + a - b, c_0) + a\mathbb{Z}$ for all $n \geq 0$. Define $\mathbf{x} = (\mathbf{x}_t(\lambda))_{\lambda \in b\mathbb{Z}}$ by $\mathbf{x}_t(\lambda) = 1$ if $\lambda = (R_{a,b,c})^n(t) - t$ for some nonnegative

integer n , and $\mathbf{x}_t(\lambda) = 0$ otherwise. Then $\mathbf{x}_t \in \mathcal{B}_b$ as $(R_{a,b,c})^n(t) - t \in b\mathbb{Z}$. Following the argument in Proposition 4.3, we have that

$$(C.3) \quad \mathbf{M}_{a,b,c}(t)\mathbf{x}_t(\mu) = 1 \quad \text{for all } 0 \leq \mu \in a\mathbb{Z}.$$

Similar to the index $Q_{a,b,c}$ in (3.6), we define

$$\tilde{Q}_{a,b,c} := \sup_{t \in \mathbb{R}} \sup_{\mathbf{x} \in \mathcal{B}_b} \tilde{Q}_{a,b,c}(t, \mathbf{x})$$

where $\tilde{K}(t, \mathbf{x}) = \{\mu \in a\mathbb{Z} : \mathbf{M}_{a,b,c}(t)\mathbf{x}(\mu) = 1\}$ and

$$\tilde{Q}(t, \mathbf{x}) = \begin{cases} 0 & \text{if } \tilde{K}(t, \mathbf{x}) = \emptyset \\ \sup\{n \in \mathbb{N} \mid [\mu, \mu + na) \subset \tilde{K}(t, \mathbf{x}) \\ \text{for some } \mu \in a\mathbb{Z}\} & \text{otherwise.} \end{cases}$$

Following the argument in Lemma 3.4, we can establish the following equivalence:

$$(C.4) \quad \mathcal{S}_{a,b,c} = \emptyset \text{ if and only if } \tilde{Q}_{a,b,c} < \infty.$$

Therefore combining (C.3) and (C.4) leads to the conclusion that $\mathcal{S}_{a,b,c} \neq \emptyset$, which is a contradiction.

(ii) \implies (i) We examine two cases $a/b \in \mathbb{Q}$ and $a/b \notin \mathbb{Q}$ to verify the empty-set property for $\mathcal{S}_{a,b,c}$.

Case 1: $a/b \in \mathbb{Q}$.

Suppose, on the contrary, that $\mathcal{S}_{a,b,c} \neq \emptyset$. Write $a/b = p/q$ for some coprime integers p and q , take $t \in \mathcal{S}_{a,b,c}$, and define $t_k = (R_{a,b,c})^k(t)$, $k \geq 0$. Then $t_k \notin [c_0 + a - b, c_0) + a\mathbb{Z}$ for all $k \geq 0$ by Proposition 4.3. Observe that for any $k \geq 0$, there exists an integer $0 \leq r_k < p$ such that $t_k - t - \frac{r_k b}{q} \in a\mathbb{Z}$ as $(t_k - t)/b \in \mathbb{Z}$ and $a = (p/q)b$. Therefore there exist two integers $n_1 < n_2$ such that $t_{n_1} - t_{n_2} \in a\mathbb{Z}$, which together with the one-to-one correspondence of the transformation $R_{a,b,c}$ on $\mathcal{S}_{a,b,c}$ implies that $t_{n_2-n_1} - t_0 \in a\mathbb{Z}$. Thus

$$(C.5) \quad \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} f((R_{a,b,c})^k(t_{n_1}))}{n} = \frac{\sum_{k=0}^{n_2-n_1-1} f((R_{a,b,c})^k(t_0))}{n_2 - n_1},$$

and the limit in (C.1) does not hold for any continuous periodic function f that it is positive on $[c_0 + a - b - \epsilon, c_0)$ and take zero value on $[0, c_0 + a - b - \epsilon)$ and $[c_0, a)$, where $\epsilon > 0$ is sufficiently small. This is a contradiction.

Case 2: $a/b \notin \mathbb{Q}$.

Take $t_0 \in \mathbb{R}$, and let $t_\infty \in [c_0 + a - b, c_0)$ such that

$$(C.6) \quad \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} f((R_{a,b,c})^k(t_0))}{n} = f(t_\infty)$$

for all continuous periodic function f with period a . We exam three subcases to verify that $t_0 \notin \mathcal{S}_{a,b,c}$.

Case 2a: $t_\infty \in (c_0 + a - b, c_0)$.

Take a continuous periodic function f such that it is positive on $(c_0 + a - b, c_0)$ and take zero value on $[0, c_0 + a - b)$ and $[c_0, a)$. Then it follows that (C.6) that $f((R_{a,b,c})^{k_0}(t_0)) > 0$ for some $k_0 \geq 0$, which implies that $(R_{a,b,c})^{k_0}(t_0)$ belongs to the black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$ of the transformation $R_{a,b,c}$. Hence $t_0 \notin \mathcal{S}_{a,b,c}$ because $\mathcal{S}_{a,b,c}$ is invariant on the transformation $R_{a,b,c}$, and $\mathcal{S}_{a,b,c}$ has empty intersection with its black hole $[c_0 + a - b, c_0) + a\mathbb{Z}$.

Case 2b: $t_\infty = c_0 + a - b$ and $(R_{a,b,c})^{k_0}(t_0) \in [c_0 + a - b + a, c_0) + a\mathbb{Z}$ for some nonnegative integer k_0 .

In this case, $t_0 \notin \mathcal{S}_{a,b,c}$ as $\mathcal{S}_{a,b,c}$ is invariant on the transformation $R_{a,b,c}$ and $[c_0 + a - b, c_0) \cap \mathcal{S}_{a,b,c} = \emptyset$.

Case 2c: $t_\infty = c_0 + a - b$ and $t_k := (R_{a,b,c})^k(t_0) \notin [c_0 + a - b, c_0) + a\mathbb{Z}$ for all nonnegative $k \geq 0$.

Take a sufficiently small positive number $\epsilon > 0$ and a periodic function $f_\epsilon(t)$ whose restriction on $[0, a)$ is given by $\max(0, 1 - |t - c_0 + a - b|/\epsilon)$. The function f_ϵ is the hat function supported in $[-\epsilon, \epsilon] + c_0 + a - b + a\mathbb{Z}$. By (C.6), for any given L there exists an integer k such that $t_{k+l} \in (c_0 + b - a - \epsilon, c_0 + b - a) + a\mathbb{Z}$ for all $0 \leq l \leq L$, as otherwise

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} f((R_{a,b,c})^k(t))}{n} \leq \frac{L-1}{L} \neq 1 = f(t_\infty).$$

As $t_{k+l} \in (c_0 + b - a - \epsilon, c_0 + b - a) + a\mathbb{Z}$ for all $0 \leq l \leq L$, we have that $t_{k+l} = (R_{a,b,c})^l(t_k) = t_k + l(\lfloor c/b \rfloor + 1)b$, $0 \leq l \leq L$. Thus $l(\lfloor c/b \rfloor + 1)b \in (-\epsilon, \epsilon) + a\mathbb{Z}$ for all $0 \leq l \leq L$, which is a contradiction as L can be chosen sufficiently large and $a/b \notin \mathbb{Q}$. \square

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